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Regularity theory for nonlocal operators

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김민현

Abstract

Regularity theory for nonlocal operators

Minhyun Kim
Department of Mathematical Sciences
The Graduate School
Seoul National University

Nonlocal operators are of significant interest in both analysis and probability theory. The thesis consists of four papers concerning interior and boundary regularity properties for nonlocal operators. The first and the second papers discuss the Krylov–Safonov theory and the Evans–Krylov and Schauder theories, respectively, for fully nonlinear nonlocal operators with rough kernels of variable orders. The interior regularity results, such as the Aleksandrov–Bakelman–Pucci estimates, Harnack inequality, Hölder estimates, and generalized Hölder estimates are established. The third paper studies the pointwise Green function estimates for a large class of nonlocal operators using purely analytic methods. In all three papers, the essence of the results is the robustness of the regularity estimates, which makes the theories for local and nonlocal operators unified.

On the other hand, the last paper deals with the boundary regularity estimates for linear nonlocal operators with kernels of variable orders. The nontrivial behaviors of the solution to the Dirichlet problem near the boundary are captured by means of the renewal function.

Key words: regularity, nonlocal operator, stochastic process
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Contents

Chapter 1

Introduction

A nonlocal operator is a mapping from functions to functions such that the information about the input function is required not only in a neighborhood of a given point but also outside of it in order to compute the value of the output function at that point. This is in contrast with local operators such as differential operators. Moreover, nonlocal operators are closely related to discontinuous stochastic models with jumps whereas local operators are associated with continuous stochastic models. It has been found that, under certain circumstances, discontinuous models are more suitable to describe natural phenomena. For instances, jump processes are currently prominent in the field of financial mathematics.

Nonlocal operators are of significant interest in both probability theory and analysis. There exists a close connection between these two fields, which motivates research in both of them. Both fields exhibit advantages in the analysis of nonlocal operators. For example, probability theory affords the use of semigroup properties, potential operators, and other probabilistic tools to analyze nonlocal operators, whereas analytical methods enable the use of nonlinear operators and robust estimates for the same purpose. In this thesis, the regularity properties of nonlocal operators are examined from the perspectives of the theory of stochastic processes and theory of partial differential (and integro-differential) equations.

The simplest example of a nonlocal operator is the fractional Laplacian

$$-(-\Delta)^{\sigma/2}u(x) = C(n,\sigma)\mathrm{P.V.}\int_{\mathbb{R}^n} \frac{u(y)-u(x)}{|y-x|^{n+\sigma}}\,\mathrm{d}y,$$

where $\sigma \in (0,2)$ and

$$C(n,\sigma) = \left(\int_{\mathbb{R}^n} \frac{1-\cos y_1}{|y|^{n+\sigma}}\,\mathrm{d}y\right)^{-1} = \frac{2^\sigma \Gamma((n+\sigma)/2)}{\pi^{n/2}|\Gamma(-\sigma/2)|}, \tag{1.0.1}$$

which is the infinitesimal generator of a rotationally symmetric σ-stable process. The nonlocality of the fractional Laplacian is reflected in its integral definition because u is required to be known over the whole domain $\mathbb{R}^n$ in order to evaluate $-(-\Delta)^{\sigma/2}u(x)$ at any point x in $\mathbb{R}^n$. Partial differential equations and integro-differential equations involving the fractional Laplacian arise in various contexts, such as continuum mechanics, population dynamics, stochastic control theory, and game theory. They have been studied extensively owing to the importance of nonlocal operators in present-day fields, and are currently well understood.

There exists a large family of Lévy processes, known as subordinate Brownian motions, that contains numerous interesting examples of nonlocal operators, such as sums of symmetric stable processes, relativistic stable processes, and geometric stable processes. As their infinitesimal generators possess kernels of variable orders, they exhibit qualitatively different behaviors compared to the rotationally symmetric σ-stable process. Over the past two decades, certain regularity properties, such as the Harnack inequality, Hölder estimates, heat kernel estimates, and Green function estimates, have been studied from the probabilistic perspective to cover several stochastic processes.

In this regard, the class of nonlocal operators for which certain regularity theories are applicable are enlarged in this thesis. The Krylov–Safonov theory, Evans–Krylov theory, and Schauder theory for nonlinear nonlocal operators with kernels of variable orders are first established using purely analytic methods. The operators considered in these theories are non-divergence form operators. On the other hand, operators in divergence form are also considered herein, and the De Giorgi–Nash–Moser theory is studied to obtain estimates for the Green function. The result reveals some interesting examples of kernels. Finally, the thesis is concluded with boundary regularity estimates for linear nonlocal operators with kernels of variable orders from the probabilistic perspective.

1.1 Probabilistic Point of View

As previously explained in the introduction, nonlocal operators can be understood as infinitesimal generators of stochastic processes which encode a great deal of information about the processes. By investigating Lévy processes and their infinitesimal generators, we may benefit from them in the analysis of nonlocal operators (see Chapter 6).

Let us observe how nonlocal operators are related to stochastic processes. We consider a Lévy process $X = (X_t, \mathbb{P}^x, t \geq 0, x \in \mathbb{R}^n)$ in $\mathbb{R}^n$ defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^x)$ with $\mathbb{P}^x(X_0 = x) = 1$. For the precise definition of Lévy process, the reader may consult [71]. From the Lévy–Khintchine formula for the characteristic functions of random variables, we have

$$\mathbb{E}^0[e^{iz\cdot X_t}] = e^{-t\Phi(z)}, \quad t \geq 0, z \in \mathbb{R}^n,$$

with the characteristic exponent

$$\Phi(z) = \frac{1}{2} z \cdot Uz + i\gamma \cdot z + \int_{\mathbb{R}^n \setminus \{0\}} \left(1 - e^{iz\cdot x} + iz \cdot x \mathbf{1}_{\{|x| \leq 1\}}\right) J(\mathrm{d}x),$$

where $U = (U_{ij})$ is an $n \times n$ symmetric nonnegative-definite matrix, $\gamma \in \mathbb{R}^n$, and μ is a measure on $\mathbb{R}^n \setminus \{0\}$ satisfying

$$\int_{\mathbb{R}^n \setminus \{0\}} \left(1 \wedge |x|^2\right) J(\mathrm{d}x) < +\infty.$$

Let us define a *transition semigroup* $(P_t)_{t\geq 0}$ *for* X by $P_t f(x) := \mathbb{E}^x[f(X_t)]$. Then an *infinitesimal generator* A *of* X is defined by

$$Au(x) := \lim_{t\searrow 0} \frac{P_t u(x) - u(x)}{t},$$

provided that the limit exists. It is well-known [77] that Au is well-defined for bounded C^2 functions u and represented by

$$\begin{aligned} Au(x) = \frac{1}{2}\sum_{i,j=1}^{n} U_{ij}\partial_{ij}u(x) + \sum_{i=1}^{n}\gamma_i\partial_i u(x) \\ + \int_{\mathbb{R}^n\setminus\{0\}} \big(u(x+y) - u(x) - \mathbf{1}_{\{|y|\leq 1\}} y\cdot\nabla u(x)\big)\, J(\mathrm{d}y). \end{aligned}$$

We call U a *diffusion coefficient of* X, γ a *linear coefficient of* X, and J a *Lévy measure of* X. Since we focus on pure jump Lévy processes, we will always assume that $U = 0$ and $\gamma = 0$.

A process under consideration within the thesis is an isotropic unimodal pure jump Lévy process with an infinite Lévy measure. In this case, $J(\mathrm{d}y)$ is an infinite measure with an isotropic density $J(|y|)$ which is non-increasing when viewed as a one-dimensional function. The infinitesimal generator can be rewritten as

$$Au(x) = \frac{1}{2}\int_{\mathbb{R}^n\setminus\{0\}} (u(x+y) + u(x-y) - 2u(x))\, J(|y|)\,\mathrm{d}y,$$

or equivalently,

$$Au(x) = \frac{1}{2}\int_{\mathbb{R}^n\setminus\{0\}} (u(x+y) + u(x-y) - 2u(x))\, \frac{J(1)}{|y|^n\varphi(|y|)}\,\mathrm{d}y, \qquad (1.1.1)$$

where φ is a function defined by

$$\varphi(r) = \frac{J(1)}{J(r)} r^{-n}. \qquad (1.1.2)$$

Note that when φ is homogeneous of degree $\sigma \in (0,2)$, X is the rotationally symmetric σ-stable process and $A = -(-\Delta)^{\sigma/2}$ is the fractional Laplacian. However, more general class of operators A with non-homogeneous functions φ will be covered in the regularity theory by imposing the so-called weak scaling condition on φ.

To motivate the study on non-homogeneous operators and on weak scaling properties, let us consider subordinate Brownian motions that contains interesting examples of processes having variable orders. We call $S = (S_t)_{t\geq 0}$ a *subordinator* if it is a Lévy process in $\mathbb{R}$ which only takes nonnegative values. Then its Laplace exponent ϕ is given by the formula

$$\mathbb{E}[e^{-\lambda S_t}] = e^{-t\phi(\lambda)}, \quad t \geq 0, \lambda > 0,$$

and it is a Bernstein function with $\lim_{\lambda \searrow 0} \phi(\lambda) = 0$, where we mean by a *Bernstein function* a nonnegative smooth function ϕ satisfying

$$(-1)^{n+1}\phi^{(n)}(\lambda) \geq 0 \quad \text{for all } n \in \mathbb{N} \text{ and } \lambda > 0.$$

It is known [72] that every Bernstein function can be uniquely represented by

$$\phi(\lambda) = b\lambda + \int_0^\infty \left(1 - e^{-\lambda x}\right) \nu(\mathrm{d}x) \tag{1.1.3}$$

with a *drift* $b \geq 0$ and a measure ν on $(0, \infty)$ satisfying $\int_0^\infty (1 \wedge x)\nu(\mathrm{d}x) < +\infty$. Conversely, every function ϕ given by (1.1.3) is a Bernstein function.

A *subordinate Brownian motion* $Y = (Y_t)_{t\geq 0} = (B_{S_t})_{t\geq 0}$ in $\mathbb{R}^n$ is a Lévy process obtained by replacing the time parameter of the Brownian motion B in $\mathbb{R}^n$ by an independent subordinator S. Then, the characteristic exponent of Y is given by $z \mapsto \phi(|z|^2)$, and the Lévy measure of Y has a density $y \mapsto j(|y|)$, where $j : (0, +\infty) \to (0, +\infty)$ is a function given by

$$j(r) = \int_0^\infty (4\pi t)^{-n/2} e^{-r^2/(4t)}\, \nu(\mathrm{d}t).$$

Moreover, we have

$$\phi(|z|^2) = \int_{\mathbb{R}^n\setminus\{0\}} (1-\cos(z\cdot y))j(|y|)\,\mathrm{d}y.$$

By [77, Section 4.1], the infinitesimal generator of Y is represented by

$$\begin{aligned} Au(x) &= -\phi(-\Delta)u(x) \\ &= \int_{\mathbb{R}^n\setminus\{0\}} \big(u(x+y)-u(x)-\mathbf{1}_{\{|y|\le 1\}}y\cdot\nabla u(x)\big)\, j(|y|)\,\mathrm{d}y \\ &= \frac{1}{2}\int_{\mathbb{R}^n\setminus\{0\}} (u(x+y)+u(x-y)-2u(x))\, j(|y|)\,\mathrm{d}y \end{aligned}$$

for any bounded C^2 functions u.

Let us provide important examples of subordinate Brownian motions, which are governed by subordinators.

Example 1.1.1. (i) The rotationally symmetric σ-stable process is the simplest example of subordinate Brownian motion with a subordinator $\phi(\lambda)=\lambda^{\sigma/2}$. In this case, we have $j(r)=C(n,\sigma)r^{-n-\sigma}$ and $-\phi(-\Delta) = -(-\Delta)^{\sigma/2}$.

(ii) An independent sum of a σ_1-stable process and a σ_2-stable process corresponds to a subordinator $\phi(\lambda)=\lambda^{\sigma_1/2}+\lambda^{\sigma_2/2}$ with $0<\sigma_1\le\sigma_2<2$.

(iii) A symmetric relativistic σ-stable process arises in the study of mathematical physics due to its application to relativistic quantum mechanics. This process is described by a subordinator $\phi(\lambda)=(\lambda+m^{2/\sigma})^{\sigma/2}-m$, where $m>0$ denotes a mass.

(iv) A process with a subordinator $\phi(r)=r^{\sigma_1/2}(\log(1+r))^{(\sigma_2-\sigma_1)/2}$ or $\phi(r)=r^{\sigma_2/2}(\log(1+r))^{(\sigma_1-\sigma_2)/2}$ for $0<\sigma_1\le\sigma_2<2$ is also an interesting example of subordinate Brownian motion.

A characteristic property that every subordinator in Example 1.1.1 enjoys is a weak scaling property. That is, there is a constant $a \geq 1$ such that

$$a^{-1}\left(\frac{R}{r}\right)^{\sigma_1/2} \leq \frac{\phi(R)}{\phi(r)} \leq a\left(\frac{R}{r}\right)^{\sigma_2/2}, \quad \text{for all } 0 < r \leq R. \tag{1.1.4}$$

Therefore, the class of subordinate Brownian motions with the weak scaling property (1.1.4) is very important in describing natural, physical, and financial phenomenon.

More generally, we may consider an isotropic unimodal pure jump Lévy process with an infinite Lévy measure, and impose the weak scaling condition to the characteristic exponent Φ. For example, we may assume that

$$a^{-1}\left(\frac{R}{r}\right)^{\sigma_1} \leq \frac{\Phi(R)}{\Phi(r)} \leq a\left(\frac{R}{r}\right)^{\sigma_2} \quad \text{for all } 0 < r \leq R,$$

for some constant $a \geq 1$. It is known [9] that $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ with a comparison constant depending only on n, where φ is a function given by (1.1.2), or by $\varphi(r) = \frac{j(1)}{j(r)}r^{-n}$ in the case of subordinate Brownian motions. Therefore, φ satisfies

$$a_1^{-1}\left(\frac{R}{r}\right)^{\sigma_1} \leq \frac{\varphi(R)}{\varphi(r)} \leq a_1\left(\frac{R}{r}\right)^{\sigma_2} \quad \text{for all } 0 < r \leq R,$$

for some constant $a_1 = a_1(a, n) \geq 1$.

The main advantage of the probabilistic approach is that we can make use of the semigroup property, heat kernel estimates, potential operators, and other probabilistic tools. Moreover, the renewal function together with these tools will play a fundamental role in capturing boundary behavior of solutions to nonlocal equations (see Chapter 6).

1.2 Analytic Point of View

Nonlocal operators have also been studied as integro-differential operators via the Fourier transform theory and PDE theory. There are two important features of an analytic point of view: on one hand, theories of second order differential equations and integro-differential equations can be unified. Motivated by the fact that the Laplacian Δ can be viewed as a limit of the fractional Laplacian $-(-\Delta)^{\sigma/2}$, the regularity estimates for nonlocal operators of order $\sigma \in (0, 2)$, that are robust in the sense that the constants in estimates do not blow up and stay uniform as $\sigma \to 2$, have attracted attention. On the other hand, PDE approaches enable us to generalize the regularity theory to nonlinear nonlocal equations. Within the thesis, the regularity theory will be established not only for linear equations, but also for nonlinear equations.

In the robust estimates, the constant $C(n, \sigma)$ given by (1.0.1) plays a fundamental role. This constant is obtained when the fractional Laplacian $(-\Delta)^{\sigma/2}$ is viewed as a pseudo-differential operator of symbol $|\xi|^\sigma$. That is, $C(n, \sigma)$ is chosen so that

$$\mathscr{F}\left((-\Delta)^{\sigma/2}u\right)(\xi) = |\xi|^\sigma \mathscr{F}u(\xi), \quad \xi \in \mathbb{R}^n.$$

While the order of differentiability of the fractional Laplacian is a single number σ, the orders of operators under consideration in this thesis cannot be characterized by a single number. The constants $C(n, \sigma_1)$ or $C(n, \sigma_2)$ are not appropriate for the robust estimates in this framework because they do not contain all information about the kernels of variable orders. Thus, we need a generalized constant that contains full information of operators.

Let us consider an operator

$$Lu(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^n \varphi(|y|)} \, \mathrm{d}y.$$

By taking the Fourier transform, we obtain

$$\begin{aligned}\mathscr{F}(-Lu)(\xi) &= -\frac{1}{2}\int_{\mathbb{R}^n} \frac{\mathscr{F}(u(\cdot+y)+u(\cdot-y)-2u(\cdot))(\xi)}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\ &= -\frac{1}{2}\left(\int_{\mathbb{R}^n} \frac{e^{i\xi\cdot y}+e^{-i\xi\cdot y}-2}{|y|^n\varphi(|y|)}\,\mathrm{d}y\right)(\mathscr{F}u)(\xi) \\ &= \left(\int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot y)}{|y|^n\varphi(|y|)}\,\mathrm{d}y\right)(\mathscr{F}u)(\xi).\end{aligned}$$

Since the function

$$\xi \mapsto \int_{\mathbb{R}^n} \frac{1-\cos(\xi\cdot y)}{|y|^n\varphi(|y|)}\,\mathrm{d}y$$

is rotationally symmetric, we have

$$\mathscr{F}(-Lu)(\xi) = \left(\int_{\mathbb{R}^n} \frac{1-\cos(|\xi|y_1)}{|y|^n\varphi(|y|)}\,\mathrm{d}y\right)(\mathscr{F}u)(\xi). \tag{1.2.1}$$

Recall that when $\varphi(r)=r^\sigma$, the integral in (1.2.1) is represented as

$$\int_{\mathbb{R}^n} \frac{1-\cos(|\xi|y_1)}{|y|^{n+\sigma}}\,\mathrm{d}y = \left(\int_{\mathbb{R}^n} \frac{1-\cos y_1}{|y|^n\varphi(|y|)}\,\mathrm{d}y\right)|\xi|^\sigma = C(n,\sigma)^{-1}|\xi|^\sigma,$$

which shows how the normalizing constant $C(n,\sigma)$ is chosen for the case of the fractional Laplacian. As generalizations of $C(n,\sigma)$ and $-(-\Delta)^\sigma$, we define the generalized constant

$$C(n,\varphi) = \left(\int_{\mathbb{R}^n} \frac{1-\cos y_1}{|y|^n\varphi(|y|)}\,\mathrm{d}y\right)^{-1}$$

and the operator

$$L_\varphi u(x) := \frac{1}{2}C(n,\varphi)\int_{\mathbb{R}^n} \frac{u(x+y)+u(x-y)-2u(x)}{|y|^n\varphi(|y|)}\,\mathrm{d}y. \tag{1.2.2}$$

In Appendix A, asymptotic properties of the constant $C(n,\varphi)$ and the operator L_φ are provided. Moreover, in the regularity theory, we will see that the generalized constant $C(n,\varphi)$ plays an important role in robust estimates for nonlocal operators with kernels of variable orders as $C(n,\sigma)$ does in the case of the fractional Laplacian.

We will also consider nonlinear operators such as

$$\mathcal{I}u(x) = \sup_{\alpha} L_\alpha u(x) \quad \text{and} \quad \mathcal{I}u(x) = \inf_{\beta} \sup_{\alpha} L_{\alpha\beta} u(x), \tag{1.2.3}$$

where $\{L_\alpha\}$ and $\{L_{\alpha\beta}\}$ are families of linear nonlocal operators. Nonlinear operators of the form (1.2.3) arise in the stochastic control theory and the game theory [78]. A characteristic property of these operators is that

$$\inf_{\alpha,\beta} L_{\alpha\beta}(u-v)(x) \le \mathcal{I}u(x) - \mathcal{I}v(x) \le \sup_{\alpha,\beta} L_{\alpha\beta}(u-v)(x).$$

Caffarelli and Silvsetre [15] introduced the concept of ellipticity for general nonlinear operators by generalizing this property. We adopt the concept and define a nonlinear elliptic operator as follows. Let $\mathcal{L}$ be a class of linear nonlocal operators of the form

$$Lu(x) = \int_{\mathbb{R}^n} \delta(u,x,y) K(y)\,\mathrm{d}y, \tag{1.2.4}$$

where $\delta(u,x,y) = u(x+y) + u(x-y) - 2u(x)$ is the second order incremental quotient. We assume that the supremum of all kernels of operators $L \in \mathcal{L}$ satisfies the integrability condition

$$\int_{\mathbb{R}^n} \left(1 \wedge |y|^2\right) \sup_{\mathcal{L}} K(y)\,\mathrm{d}y < +\infty. \tag{1.2.5}$$

The *maximal operator* and the *minimal operator* with respect to $\mathcal{L}$ are defined as

$$\mathcal{M}^+_{\mathcal{L}} u(x) = \sup_{L\in\mathcal{L}} Lu(x) \quad \text{and} \quad \mathcal{M}^-_{\mathcal{L}} u(x) = \inf_{L\in\mathcal{L}} Lu(x).$$

Using these extremal operators, we define a fully nonlinear elliptic integro-differential operator. We say that a function u is of $C^{1,1}$ at a point x, and denote by $u \in C^{1,1}(x)$, if there are a vector $v \in \mathbb{R}^n$ and a number $M > 0$

such that

$$|u(x+y) - u(x) - v \cdot y| \le M|y|^2 \quad \text{for } |y| \text{ sufficiently small.}$$

We say that a function u is $C^{1,1}$ in a set Ω if $u \in C^{1,1}(x)$ for all $x \in \Omega$ with a uniform constant M.

Definition 1.2.1. Let $\mathcal{L}$ be a class of linear nonlocal operators of the form (1.2.4) satisfying the integrability condition (1.2.5). An *elliptic operator* $\mathcal{I}$ *with respect to* $\mathcal{L}$ is an operator with the following properties:

(i) If u is bounded in $\mathbb{R}^n$ and is of $C^{1,1}(x)$, then $\mathcal{I}(u, x)$ is defined classically.

(ii) If u is bounded in $\mathbb{R}^n$ and is C^2 in some open set Ω, then $\mathcal{I}(u, x)$ is a continuous function in Ω.

(iii) If u and v are bounded in $\mathbb{R}^n$ and are of $C^{1,1}(x)$, then

$$\mathcal{M}^-_{\mathcal{L}}(u-v)(x) \le \mathcal{I}(u, x) - \mathcal{I}(v, x) \le \mathcal{M}^-_{\mathcal{L}}(u-v)(x).$$

Notice that Definition 1.2.1 is given in full generality so that the definition covers non-translation invariant fully nonlinear operators. When we deal with a translation invariant operator, we will write $\mathcal{I}u(x)$ instead of $\mathcal{I}(u, x)$.

Let us introduce some classes that we are mainly concerned with throughout the thesis. The most important one is the class $\mathcal{L}_0(\varphi)$ of linear nonlocal operators of the form (1.2.4) with measurable kernels K satisfying

$$\lambda \frac{C(n,\varphi)}{|y|^n \varphi(|y|)} \le K(y) \le \Lambda \frac{C(n,\varphi)}{|y|^n \varphi(|y|)} \tag{1.2.6}$$

with ellipticity constants $0 < \lambda \le \Lambda$. The extremal operators with respect to

$\mathcal{L}_0(\varphi)$ are represented by

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u(x) = C(n,\varphi)\int_{\mathbb{R}^n} \frac{\Lambda\delta_+(u,x,y) - \lambda\delta_-(u,x,y)}{|y|^n\varphi(|y|)}\,\mathrm{d}y \quad \text{and}$$
$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)}u(x) = C(n,\varphi)\int_{\mathbb{R}^n} \frac{\lambda\delta_+(u,x,y) - \Lambda\delta_-(u,x,y)}{|y|^n\varphi(|y|)}\,\mathrm{d}y.$$

This class is a generalization of the class $\mathcal{L}_0(\sigma)$ that was first introduced in [15]. We will establish the Krylov–Safonov, Evans–Krylov, and Schauder theories for fully nonlinear operators with respect to the class $\mathcal{L}_0(\varphi)$ or other classes of linear operators with more regular kernels. See Chapter 3 for the class $\mathcal{L}_1(\varphi)$ and Chapter 4 for the class $\mathcal{L}_\psi(\varphi)$.

1.3 Results

In this thesis, the interior and boundary regularities of solutions to linear and nonlinear nonlocal elliptic equations are developed using analytic and probabilistic methods. The first three chapters, i.e., Chapter 3–Chapter 5, discuss interior regularity estimates, and the last chapter, i.e., Chapter 6, deals with boundary regularity estimates.

We begin with the Krylov–Safonov theory for fully nonlinear nonlocal operators with kernels of variable orders in Chapter 3. A nonlocal version of Aleksandrov–Bakelman–Pucci estimates, Harnack inequality, and interior Hölder estimates for viscosity solutions to the equation $\mathcal{I}(u,x) = f(x)$, where $\mathcal{I}$ denotes an elliptic operator with respect to $\mathcal{L}_0(\varphi)$, will be established. The main feature of these results is the robustness of the regularity estimates. Contrary to the case of stable-like operators, the operators considered have kernels of variable orders, meaning that φ can oscillate between two functions r^{σ_1} and r^{σ_2}. To capture the correct scale and obtain robust estimates, the generalized constant $C(n,\varphi)$ and appropriate scale functions must be considered. Moreover, the $C^{1,\alpha}$ regularity of viscosity solutions is established, provided that operator $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_1(\varphi)$.

In Chapter 4, we discuss the regularity theory associated with higher-order regularity, such as the $C^{\varphi\psi}$ regularity as a generalization of the $C^{\sigma+\alpha}$ regularity. Precisely, the Evans–Krylov-type interior estimates for concave translation invariant nonlocal fully nonlinear equations with respect to $\mathcal{L}_0(\varphi)$ and the Schauder-type interior estimates for equations with x dependence in a generalized Hölder fashion are established. We indicate that, similar to the Krylov–Safonov theory, all the estimates in Chapter 4 are robust.

Because of the non-homogeneity of function φ, the equations are not scale invariant, violating the standard blowup sequence argument. This is because the rescaled equations and rescaled solutions are related to new scale functions at each scale. We overcome this difficulty by observing that the rescaled equations belong to the same class of equations with the same constants in the weak scaling condition (although they are associated with different scale functions) and obtaining uniform estimates that only depend on these constants.

While the operators considered in Chapter 3 and Chapter 4 are non-divergence form operators, those considered in Chapter 5 are divergence form operators. In Chapter 5, we study Green functions for linear nonlocal operators of divergence form. The existence, uniqueness, symmetry, and pointwise upper and lower bounds of the Green functions for nonlocal operators are treated via the De Giorgi–Nash–Moser theory.

The results in this chapter are based on the fact that the pointwise estimates of Green functions for nonlocal operators have not been established thus far from an analytic perspective, whereas these have been studied extensively from a probabilistic perspective. In this chapter, by using purely analytic methods, we could obtain robust estimates that were not provided by probabilistic approaches. Our result is novel not only because the methods are purely analytic but also because it covers a new interesting example of an operator whose heat kernel estimates do not hold so that Green function estimates cannot be obtained by simply integrating the heat kernel. This shows the averaging effect of Green functions. Because this example is irrelevant to the operators that have kernels of variable orders, the operators in this chapter are assumed to have kernels of fixed order to capture the essence of the results.

Let us consider the regularity properties of viscosity solutions up to the boundary. In Chapter 6, viscosity solutions to the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases}$$

where D denotes a bounded $C^{1,1}$ open set in $\mathbb{R}^n$ and L a linear operator with kernel of variable orders, are studied. The solvability of the Dirichlet problem and generalized Hölder estimates up to the boundary will be established. However, the difficulty arises in capturing the behavior of solutions near the boundary because the kernels of the operators are of variable orders. A simple barrier function $\mathrm{dist}(x, \mathbb{R}^n \setminus D)^{\sigma/2}$ that is used in the case of the fractional Laplacian does not work in our framework. In the analysis of the boundary behavior of solutions, we benefit from the probability theory. The so-called renewal function V will be introduced to replace the role of polynomial $x^{\sigma/2}$ and obtain global C^V estimates of the solutions. Moreover, the Hölder regularity of the quotient $u/V(\mathrm{dist}(x, \mathbb{R}^n \setminus D))$ is established using the Krylov boundary Harnack method.

1.4 Notations

Unless otherwise specified, we use the following notations throughout the thesis.

We denote by $\mathbb{R}^n$ the n-dimensional Euclidean space, endowed with the Lebesgue measure $\mathrm{d}x$. The upper half space $\{(x_1, \cdots, x_n) \in \mathbb{R}^n : x_n > 0\}$ is denoted by $\mathbb{R}^n_+$. We denote the ball in $\mathbb{R}^n$ with center $x \in \mathbb{R}^n$ and radius $R > 0$ by $B_R(x)$ or $B(x, R)$, and write $B_R(0) = B_R$ in the case $x = 0$. The volume of unit ball is denoted by ω_n. The cube in $\mathbb{R}^n$ whose sides are parallel to axes and have length R is denoted by Q_R. Every cube within the thesis is centered at the origin. The region where equations are satisfied is usually denoted by Ω, but D will also be used in Chapter 6 since Ω stands for a sample space in the probability theory.

We denote $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. For a function $u : \Omega \to \mathbb{R}$, we write $u_+ := u \vee 0$ and $u_- := -(u \wedge 0)$, so that $u = u_+ - u_-$. For any functions $f, g : (0, \infty) \to [0, \infty)$, we write $f \asymp g$ for $r > 0$ ($0 < r \le r_0$, respectively), if there is a constant $c \ge 1$ such that $c^{-1} f(r) \le g(r) \le cf(r)$ for $r > 0$ ($0 < r \le r_0$, respectively).

Let us next recall some function spaces. For k a nonnegative integer (or ∞), $C^k(\Omega)$ is defined to be the set of functions having all derivatives of order less than or equal to k continuous in Ω, and $C^k(\overline{\Omega})$ the set of all functions in $C^k(\Omega)$ all of whose derivatives of order less than or equal to k have continuous extensions to $\overline{\Omega}$. It is to be noted that $C^0(\Omega)$ and $C^0(\overline{\Omega})$ are sets of continuous functions on Ω and $\overline{\Omega}$, respectively. In particular, the spaces $C^k(\overline{\Omega})$ are Banach spaces equipped with the norms

$$\|u\|_{C^k(\overline{\Omega})} = \|u\|_{k;\Omega} = \sum_{j=0}^{k} [u]_{j;\Omega} = \sum_{j=0}^{k} \sup_{|\gamma|=j} \sup_{\Omega} |D^\gamma u|.$$

For $\alpha \in (0, 1)$, we define the *Hölder spaces* $C^{k,\alpha}(\overline{\Omega})$ ($C^{k,\alpha}(\Omega)$, respectively) as the subspaces of $C^k(\overline{\Omega})$ ($C^k(\Omega)$, respectively) consisting of functions whose k-th order derivatives are uniformly Hölder continuous (locally Hölder continuous, respectively) with exponent α in Ω. For the sake of brevity we write $C^{0,\alpha}(\Omega) = C^\alpha(\Omega)$ and $C^{0,\alpha}(\overline{\Omega}) = C^\alpha(\overline{\Omega})$. For a non-integer $\beta > 0$, by C^β we mean the Hölder space $C^{k,\alpha}$, where $k \in \mathbb{N}$ and $\alpha \in (0, 1)$ are such that $\beta = k + \alpha$.

The Hölder spaces $C^{k,\alpha}(\overline{\Omega})$, equipped with the norms

$$\begin{aligned}\|u\|_{C^{k,\alpha}(\overline{\Omega})} &= \|u\|_{k,\alpha;\Omega} = \|u\|_{k;\Omega} + [u]_{k,\alpha;\Omega} \\ &= \|u\|_{k;\Omega} + \sup_{|\gamma|=k} \sup_{x,y\in\Omega, x\neq y} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x-y|^\alpha},\end{aligned}$$

or the non-dimensional norms

$$\|u\|'_{C^{k,\alpha}(\overline{\Omega})} = \|u\|'_{k,\alpha;\Omega} = \|u\|'_{k;\Omega} + [u]'_{k,\alpha;\Omega} = \sum_{j=0}^{k} d^j [u]_{j;\Omega} + d^{k+\alpha}[u]_{k,\alpha;\Omega},$$

where $d = \mathrm{diam}(\Omega)$, are Banach spaces. It is sometimes useful to work with interior norms

$$\begin{aligned}\|u\|^*_{C^{k,\alpha}(\Omega)} &= \|u\|^*_{k,\alpha;\Omega} = \|u\|^*_{k;\Omega} + [u]^*_{k,\alpha;\Omega} \\ &= \sum_{j=0}^{k} \sup_{|\gamma|=j} \sup_{x\in\Omega} d_x^j |D^\gamma u(x)| + \sup_{|\gamma|=k} \sup_{x,y\in\Omega, x\neq y} d_{x,y}^{k+\alpha} \frac{|D^\gamma u(x) - D^\gamma u(y)|}{|x-y|^\alpha},\end{aligned}$$

where $d_x = \mathrm{dist}(x, \partial\Omega)$ and $d_{x,y} = d_x \wedge d_y$. It is to be noted that $\|\cdot\|^*_{k,\alpha;\Omega}$ are norms on the subspaces of $C^{k,\alpha}(\Omega)$ for which they are finite. See Section 2.2 for finer scale of function spaces generalizing Hölder spaces.

We are also interested in the fractional Sobolev spaces $H^{\sigma/2}(\mathbb{R}^n)$ and related spaces that are more suitable for divergence form equations. Actually, more general spaces such as $H^\mu(\mathbb{R}^n)$ and $H^\mu_\Omega(\mathbb{R}^n)$ will be defined later. See Chapter 5 for the definition of these spaces.

By a universal constant, we mean a constant C depending only on some quantities in assumptions, such as dimension n, ellipticity constants λ and Λ, and so on, but not on solutions u or data. Throughout the thesis, universal constants C in the regularity estimates may differ from line to line.

Chapter 2

Preliminaries

2.1 Weak Scaling Condition

In this section we formulate the weak scaling condition, which will describe the behavior of kernels of operators, and analyze the behavior of the constant $C(n, \varphi)$ under the weak scaling condition. As mentioned above in Section 1.2, the constant $C(n, \varphi)$ plays a fundamental role in robust estimates as $C(n, \sigma)$ does for the case of the fractional Laplacian. By introducing some scale functions and using the weak scaling condition, we investigate behaviors of the constant $C(n, \varphi)$.

We say that a function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the *weak scaling condition* with constants $0 < \sigma_1 \le \sigma_2 < 2$ and $a \ge 1$ if

$$a^{-1} \left(\frac{R}{r} \right)^{\sigma_1} \le \frac{\varphi(R)}{\varphi(r)} \le a \left(\frac{R}{r} \right)^{\sigma_2} \quad \text{for all } 0 < r \le R. \tag{2.1.1}$$

We say that φ satisfies the *weak scaling condition at zero* with constants $0 < \sigma_1 \le \sigma_2 < 2$ and $a \ge 1$ if (2.1.1) holds only for $0 < r \le R \le 1$. Since φ is allowed to oscillate between two functions r^{σ_1} and r^{σ_2}, it is referred to as having *variable orders*.

Let us introduce scale functions $\underline{C}_\varphi, \overline{C}_\varphi : \mathbb{R}_+ \to \mathbb{R}$, which are defined by

$$\underline{C}_\varphi(R) := \int_0^R \frac{r}{\varphi(r)}\,\mathrm{d}r \quad \text{and} \quad \overline{C}_\varphi(R) := \int_R^\infty \frac{1}{r\varphi(r)}\,\mathrm{d}r. \tag{2.1.2}$$

We will drop the subscript φ in $\overline{C}_\varphi$ and $\underline{C}_\varphi$ when it is clear from the context. These functions correspond to $\frac{R^{2-\sigma}}{2-\sigma}$ and $\frac{R^{-\sigma}}{\sigma}$, respectively, in the case of the fractional Laplacian. Let us first observe that how they behave in general case.

Lemma 2.1.1. *Let φ be a function satisfying the weak scaling condition* (2.1.1) *with constants $0 < \sigma_1 \le \sigma_2 < 2$ and $a \ge 1$. Then*

$$\frac{1}{a(2-\sigma_1)}\frac{R^2}{\varphi(R)} \le \underline{C}(R) \le \frac{a}{2-\sigma_2}\frac{R^2}{\varphi(R)} \tag{2.1.3}$$

and

$$\frac{1}{a\sigma_2}\frac{1}{\varphi(R)} \le \overline{C}(R) \le \frac{a}{\sigma_1}\frac{1}{\varphi(R)} \tag{2.1.4}$$

for all $R > 0$. If φ satisfies the weak scaling condition at zero, then (2.1.3) *and* (2.1.4) *hold only for $0 < R \le 1$.*

Proof. Suppose that φ satisfies the weak scaling condition (2.1.1). For $R > 0$, we have

$$\underline{C}(R) \ge \int_0^R \frac{1}{a}\frac{r}{\varphi(R)}\left(\frac{R}{r}\right)^{\sigma_1}\mathrm{d}r = \frac{1}{a(2-\sigma_1)}\frac{R^2}{\varphi(R)},$$

which provides the first inequality in (2.1.3). The second inequality in (2.1.3) and the inequalities in (2.1.4) can be proved in the same manner. □

We remark that Lemma 2.1.1 shows that $1/\underline{C}(1)$ and $1/\overline{C}(1)$ serve as constants $2-\sigma$ and σ, respectively, for the case of the fractional Laplacian. The following lemma will also be used frequently in the sequel.

Lemma 2.1.2. *Let φ be a function satisfying the weak scaling condition* (2.1.1) *with constants $0 < \sigma_1 \le \sigma_2 < 2$ and $a \ge 1$. Then*

$$\frac{\underline{C}(R)}{\underline{C}(tR)} \le 1 + a^2 t^{-(2-\sigma_1)}$$

for all $R > 0$ *and* $t \in (0, 1)$.

Proof. A simple computation shows that

$$\begin{aligned}
\frac{\underline{C}(R)}{\underline{C}(tR)} &= 1 + \frac{1}{\underline{C}(tR)} \int_t^R \frac{\varphi(tR)}{\varphi(r)} \frac{r}{\varphi(tR)} \, \mathrm{d}r \\
&\le 1 + a(2-\sigma_1) \frac{\varphi(tR)}{(tR)^2} \int_{tR}^R a \left(\frac{tR}{r} \right)^{\sigma_1} \frac{r}{\varphi(tR)} \, \mathrm{d}r \\
&\le 1 + a^2 (tR)^{-(2-\sigma_1)} \left(R^{2-\sigma_1} - (tR)^{2-\sigma_1} \right) \le 1 + a^2 t^{-(2-\sigma_1)},
\end{aligned}$$

where we have used (2.1.1) and (2.1.3). □

Motivated by the fact that the constant $C(n, \sigma)$ for the fractional Laplacian is comparable with $\sigma(2-\sigma)$, we prove the following lemma. It will play an important role in regularity estimates that are robust.

Lemma 2.1.3. *Suppose that* φ *satisfies the weak scaling condition* (2.1.1) *with constants* $0 < \sigma_1 \le \sigma_2 < 2$ *and* $a \ge 1$*. There are constants* $c_1, c_2 > 0$*, depending only on* n *and* a*, such that*

$$\frac{c_1}{\underline{C}(R) + \overline{C}(R)} \le C(n, \varphi) \le \frac{c_2}{\underline{C}(1) + \overline{C}(1)} \tag{2.1.5}$$

for all $R > 0$.

Proof. We recall that the constant $C(n, \varphi)$ is given by

$$C(n, \varphi)^{-1} = \int_{\mathbb{R}^n} \frac{1 - \cos y_1}{|y|^n \varphi(|y|)} \, \mathrm{d}y.$$

By rewriting the above integral as a double integral, we have

$$C(n, \varphi)^{-1} = \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^n} \int_{\mathbb{R}} \frac{1 - \cos(r/\zeta)}{|r| \varphi(|r|)} \, \mathrm{d}r \, \mathrm{d}y', \tag{2.1.6}$$

where $\zeta = \zeta(y') = (1 + |y'|^2)^{1/2}$. Using the inequality $1 - \cos \frac{r}{\zeta} \le \frac{r^2}{2\zeta^2}$, we estimate the inner integral in (2.1.6) as

$$\begin{aligned}\int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi(|r|)}\,\mathrm{d}r &= 2\int_0^R \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r + 2\int_R^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r \\ &\le \frac{1}{\zeta^2}\underline{C}(R) + 4\overline{C}(R).\end{aligned}$$

Thus, the lower bound of $C(n,\varphi)$ in (2.1.5) follows from that

$$C(n,\varphi)^{-1} \le \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^n}\left(\frac{1}{\zeta^2}\underline{C}(R) + 4\overline{C}(R)\right)\mathrm{d}y' \le C\left(\underline{C}(R) + \overline{C}(R)\right),$$

with a constant C depending only on n.

For the upper bound, we first note that $1-\cos\frac{r}{\zeta} \ge \frac{r^2}{4\zeta^2}$ for $|r| \le 1$ since $\zeta = (1+|y'|)^{1/2} \ge 1$. Then we have

$$\int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi(|r|)}\,\mathrm{d}r \ge \frac{1}{2\zeta^2}\underline{C}(1) + 2\int_1^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r. \tag{2.1.7}$$

For the last term in (2.1.7), we observe that

$$\sum_{m=0}^\infty \int_{2m\zeta\pi+\frac{1}{2}\zeta\pi}^{2m\zeta\pi+\frac{3}{2}\zeta\pi} \frac{\mathrm{d}r}{r\varphi(r)} \le \int_1^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r \tag{2.1.8}$$

and that

$$\begin{aligned}\sum_{m=0}^\infty \int_{2m\zeta\pi+\frac{3}{2}\zeta\pi}^{2m\zeta\pi+\frac{5}{2}\zeta\pi} \frac{\mathrm{d}r}{r\varphi(r)} &= \sum_{m=0}^\infty \int_{2m\zeta\pi+\frac{1}{2}\zeta\pi}^{2m\zeta\pi+\frac{3}{2}\zeta\pi} \frac{\mathrm{d}r}{(r+\zeta\pi)\varphi(r+\zeta\pi)} \\ &\le a\sum_{m=0}^\infty \int_{2m\zeta\pi+\frac{1}{2}\zeta\pi}^{2m\zeta\pi+\frac{3}{2}\zeta\pi} \frac{\mathrm{d}r}{r\varphi(r)} \\ &\le a\int_1^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r,\end{aligned} \tag{2.1.9}$$

where the weak scaling condition (2.1.1) is used in the first inequality in

(2.1.9). Since we have $r - \zeta\pi \geq \frac{1}{1+\zeta\pi} r$ for $r \geq 1 + \zeta\pi$, we also obtain that

$$\begin{aligned} \int_1^{\frac{1}{2}\zeta\pi} \frac{\mathrm{d}r}{r\varphi(r)} &= \int_{1+\zeta\pi}^{\frac{3}{2}\zeta\pi} \frac{\mathrm{d}r}{(r-\zeta\pi)\varphi(r-\zeta\pi)} \\ &\leq a\,(1+\zeta\pi)^{1+\sigma_2} \int_{\frac{1}{2}\zeta\pi}^{\frac{3}{2}\zeta\pi} \frac{\mathrm{d}r}{r\varphi(r)} \\ &\leq a\,(1+\zeta\pi)^3 \int_1^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r. \end{aligned} \tag{2.1.10}$$

Combining (2.1.8)–(2.1.10), we have

$$\overline{C}(1) \leq C\zeta^3 \int_1^\infty \frac{1-\cos(r/\zeta)}{r\varphi(r)}\,\mathrm{d}r. \tag{2.1.11}$$

Therefore, by putting (2.1.11) into (2.1.7), we arrive at

$$C(n,\varphi)^{-1} \geq \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^n} \left(\frac{1}{2\zeta^2}\underline{C}(1) + \frac{C}{\zeta^3}\overline{C}(1) \right) \mathrm{d}y' \geq C\left(\underline{C}(1) + \overline{C}(1)\right),$$

which finishes the proof. □

The estimates (2.1.5) are sufficient for the regularity theory within the thesis. However, something more about asymptotic behaviors of $C(n,\varphi)$ can be investigated. In Appendix A, such asymptotic behaviors are explained.

2.2 Generalized Hölder Space

This section is devoted to the generalized Hölder spaces. We adopt the definition of generalized Hölder spaces from [2]. For more exposition of these spaces, see [2] and references therein.

In order to define the generalized Hölder spaces, let us first discuss the concept of order of differentiability. Throughout this section, we always assume that $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a function satisfying

$$\psi(1) = 1 \quad \text{and} \quad \lim_{r\to+0} \psi(r) = 0.$$

The function ψ is said to be *almost increasing* if there is a constant $c \in (0,1]$ such that $c\psi(r) \leq \psi(R)$ for all $r \leq R$. Similarly, ψ is said to be *almost decreasing* if there is a constant $C \in [1,\infty)$ such that $\psi(R) \leq C\psi(r)$ for all $r \leq R$. We adopt, from [2], the definitions of indices M_ψ and m_ψ, which are given by

$$\begin{aligned} M_\psi &= \inf\left\{\alpha \in \mathbb{R} : r \mapsto r^{-\alpha}\psi(r) \text{ is almost decreasing}\right\}, \\ m_\psi &= \sup\left\{\alpha \in \mathbb{R} : r \mapsto r^{-\alpha}\psi(r) \text{ is almost increasing}\right\}, \end{aligned}$$

and denote by I_ψ the closed interval $[m_\psi, M_\psi]$. The interval I_ψ describes the range of orders of differentiability induced by ψ. For instances, if $\psi(r) = r^\alpha$ or $\psi(r) = r^\alpha|\log(2/r)|$, then we have $M_\psi = m_\psi = \alpha$, and if $\psi(r) = r^\alpha + r^\beta$, then we have $M_\psi = \max\{\alpha,\beta\}$ and $m_\psi = \min\{\alpha,\beta\}$. We also observe that for the function φ satisfying the weak scaling property (2.1.1), we have $I_\varphi \subset [\sigma_1, \sigma_2]$. We may and do assume that $I_\varphi = [\sigma_1,\sigma_2]$ by considering the largest σ_1 and the smallest σ_2 such that (2.1.1) holds.

Let Ω be an open subset of $\mathbb{R}^n$ and let us define the generalized Hölder space.

Definition 2.2.1. Suppose that $m_\psi \in (k, k+1]$ for some nonnegative integer k. The Banach space $C^\psi(\overline{\Omega})$ is defined as the subspace of $C^k(\overline{\Omega})$, equipped with the norm

$$\|u\|_{C^\psi(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + [u]_{C^\psi(\overline{\Omega})} := \|u\|_{C^k(\overline{\Omega})} + \sup_{x,y\in\Omega, x\neq y} \frac{|D^k u(x) - D^k u(y)|}{\psi(|x-y|)|x-y|^{-k}}.$$

Let us write $\|\cdot\|_{C^\psi(\overline{\Omega})} = \|\cdot\|_{\psi;\Omega}$ and $[\,\cdot\,]_{C^\psi(\overline{\Omega})} = [\,\cdot\,]_{\psi;\Omega}$ for the sake of brevity. We will abuse notation and write $C^\psi = C^\alpha$ instead of C^{r^α} when ψ is a polynomial of power $\alpha \notin \mathbb{N}$ (If $\alpha \in \mathbb{N}$, then the space C^ψ gives the Lipschitz space $C^{0,1}$, not C^1. However we will not make use of Lipschitz spaces in the thesis). In particular, the generalized Hölder space with the modulus of continuity $\varphi(r)r^\alpha$ will be frequently used in Chapter 4, and in this case it will be denoted by $C^{\varphi+\alpha}$.

It is sometimes useful to introduce non-dimensional norms on $C^\psi(\overline{\Omega})$. If

Ω is bounded, we set

$$\|u\|'_{\psi;\Omega} := \|u\|'_{k;\Omega} + [u]'_{\psi;\Omega} := \|u\|'_{k;\Omega} + \psi(d)[u]_{\psi;\Omega}, \tag{2.2.1}$$

where $d = \operatorname{diam}\Omega$. The spaces $C^{\psi}(\overline{\Omega})$, equipped with the norm (2.2.1), are also Banach spaces. We will also make use of the interior norms

$$\|u\|^*_{\psi;\Omega} := \|u\|^*_{k;\Omega} + [u]^*_{\psi;\Omega} := \|u\|^*_{k;\Omega} + \sup_{x,y\in\Omega, x\neq y} \psi(d_{x,y}) \frac{|D^k u(x) - D^k u(y)|}{\psi(|x-y|)|x-y|^{-k}},$$

where $d_{x,y} := \min\{d_x, d_y\}$ and $d_x = \operatorname{dist}(x, \partial\Omega)$. The space of functions in $C^{\psi}(\Omega)$ whose interior norms are finite is a Banach space, equipped with the norm $\|\cdot\|^*_{\psi;\Omega}$.

The following interpolation lemma will be used frequently in the sequel. The proof is given in [2, Proposition 2.6] for the whole space, but the same proof holds for balls.

Lemma 2.2.2. *Assume $I_{\psi_1}, I_{\psi_2} \subset (0,1)\cup(1,2)\cup(2,3)$ and $M_{\psi_1} < m_{\psi_2}$, and let $\varepsilon \in (0,1)$. Then there is a constant $C = C(n, \psi_1, \psi_2, \varepsilon) > 0$ such that*

$$\|u\|'_{\psi_1;B} \le C\|u\|_{0;B} + \varepsilon\|u\|'_{\psi_2;B},$$

for every ball $B = B_R(x_0)$.

We remark that, in Lemma 2.2.2, the dependence of the constant C on ψ_1 and ψ_2 can be elaborated into the dependence on M_{ψ_1}, m_{ψ_1}, M_{ψ_2}, m_{ψ_2}, and the constants appearing in the definitions of almost increasing and almost decreasing functions. Therefore, whenever we use Lemma 2.2.2 with $\psi_1 = \varphi$, we can say that the constant C depends on σ_1, σ_2, and a (or on σ_0 and a if $\sigma_0 \le \sigma_1 \le \sigma_2 < 2$).

2.3 Viscosity Solution

Within the thesis various concepts of solutions for linear or nonlinear nonlocal operators will be discussed. As we have seen in Section 1.2, if u is bounded in $\mathbb{R}^n$ and $C^{1,1}$ in a neighborhood of a point x, then a value $Lu(x)$ or $\mathcal{I}(u,x)$ can be evaluated classically. We will consider such classical solutions in Chapter 4 even in a more general sense—when we are dealt with operators with kernels comparable to $\frac{1}{|y|^n\varphi(|y|)}$, the $C^{1,1}$ regularity near x can be weakened by $C^{\varphi+\varepsilon}$ regularity. Moreover, the concept of weak solutions, which is appropriate for operators of divergence form, will be used in Chapter 5. However, in this section let us focus on the concept of viscosity solutions for nondivegence form operators, which will be used in Chapter 3 and Chapter 6.

We adopt the definition of viscosity solutions for (translation invariant) nonlocal operators in [15]. We refer the reader to [18] for viscosity solutions of local equations.

Definition 2.3.1. Let Ω be an open subset of $\mathbb{R}^n$, and let $\mathcal{I}$ be a translation invariant elliptic operator in the sense of Definition 1.2.1. A bounded function $u : \mathbb{R}^n \to \mathbb{R}$ which is upper (lower) semicontinuous in $\overline{\Omega}$ is said to be a *viscosity subsolution* (*viscosity supersolution*) to $\mathcal{I}u = f$ in Ω, and we write $\mathcal{I}u \geq f$ ($\mathcal{I}u \leq f$) in Ω, if whenever a function $v \in C^{1,1}(x)$ touches u from above (below) at $x \in \Omega$ in a small neighborhood N of x, i.e., $v(x) = u(x)$ and $v > u$ ($v < u$) in $N \setminus \{x\}$, then the function

$$w := \begin{cases} v & \text{in } N, \\ u & \text{in } \mathbb{R}^n \setminus N, \end{cases}$$

satisfies $\mathcal{I}w(x) \geq f(x)$ ($\mathcal{I}w(x) \leq f(x)$). A function u is said to be a *viscosity solution* if u is both a viscosity subsolution and a viscosity supersolution.

For nonlinear nonlocal operators of the form (1.2.3), we have the following properties. The proofs can be found in [15, Section 3 and 4].

Lemma 2.3.2. *Let $\mathcal{I}$ be the operator of the form* (1.2.3) *with L_α or $L_{\alpha\beta} \in \mathcal{L}_0(\varphi)$. Then $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_0(\varphi)$. Moreover, if $\mathcal{I}u \geq f$ in Ω in the viscosity sense and if a function $\psi \in C^{1,1}(x)$ touches u from above at x, then $\mathcal{I}u(x)$ is defined in the classical sense and $\mathcal{I}u(x) \geq f(x)$.*

Stability properties of viscosity solutions to nonlocal equations with respect to the natural limits for lower semicontinuous functions were proved in [15]. The limit of this type is usually called a Γ-limit.

Definition 2.3.3. We say that a sequence of lower semicontinuous functions u_k Γ-*converges* to u in a set Ω if the following conditions hold:

(i) For every sequence $x_k \to x$ in Ω,

$$\liminf_{k\to\infty} u_k(x_k) \geq u(x).$$

(ii) For every $x \in \Omega$, there is a sequence $x_k \to x$ in Ω such that

$$\limsup_{k\to\infty} u_k(x_k) = u(x).$$

Note that a uniformly convergent sequence u_k also converges in the Γ sense. We refer to [15] for the proof of the following lemma.

Lemma 2.3.4. *Let $\mathcal{I}$ be elliptic in the sense of Definition 1.2.1 and u_k be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ such that*

(i) $\mathcal{I}u_k \leq f_k$ in Ω in the viscosity sense,

(ii) $u_k \to u$ in the Γ sense in Ω,

(iii) $u_k \to u$ a.e. *in $\mathbb{R}^n$,*

(iv) $f_k \to f$ locally uniformly in Ω for some continuous function f.

Then, $\mathcal{I}u \leq f$ in Ω in the viscosity sense.

Corollary 2.3.5. *Let $\mathcal{I}$ be elliptic in the sense of Definition 1.2.1 and u_k be a sequence of functions that are uniformly bounded in $\mathbb{R}^n$ such that*

(i) $\mathcal{I}u_k = f_k$ in Ω in the viscosity sense,

(ii) $u_k \to u$ locally uniformly in Ω,

(iii) $u_k \to u$ a.e. in $\mathbb{R}^n$,

(iv) $f_k \to f$ locally uniformly in Ω for some continuous function f.

Then, $\mathcal{I}u = f$ in Ω in the viscosity sense.

The stability properties of viscosity solutions are used to prove the comparison principle. In [15], the comparison principle is proved under the mild assumption on the class of operators. Let us provide the following comparison principle by checking [15, Assumption 5.1] holds true when $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_0(\varphi)$.

Theorem 2.3.6 (Comparison principle)**.** *Let $\mathcal{I}$ be an elliptic operator with respect to $\mathcal{L}_0(\varphi)$. Let $\Omega \subset \mathbb{R}^n$ be a bounded open set. If u and v are bounded functions in $\mathbb{R}^n$ such that $\mathcal{I}u \geq f$ and $\mathcal{I}v \leq f$ in Ω in the viscosity sense for some continuous function f, and if $u \leq v$ in $\mathbb{R}^n \setminus \Omega$, then $u \leq v$ in Ω.*

Proof. The proof is the same as one for [15, Theorem 5.2] if [15, Assumption 5.1] is provided. We claim that for every $R \geq 4$, there exists a constant $\delta = \delta(R) > 0$ such that $Lw_R > \delta$ in B_R for any operator $L \in \mathcal{L}_0(\varphi)$, where $w_R(x) = 1 \wedge |\frac{x}{2R}|^2$. Indeed, for $x \in B_R$ we have

$$\delta(w_R, x, y) = \frac{|x+y|^2}{4R^2} + \frac{|x-y|^2}{4R^2} - \frac{2|x|^2}{4R^2} = \frac{|y|^2}{2R^2} \quad \text{if } x \pm y \in B_{2R}$$

and

$$\delta(w_R, x, y) \geq 1 - \frac{|x|^2}{2R^2} \geq 0 \quad \text{if } x + y \notin B_{2R} \text{ or } x - y \notin B_{2R}.$$

Thus, for any operator $L \in \mathcal{L}_0(\varphi)$ we obtain

$$Lw_R(x) \geq C_\varphi \lambda \int_{B_R} \frac{|y|^2}{2R^2} \frac{\mathrm{d}y}{|y|^n \varphi(|y|)} =: \delta(R) > 0,$$

which proves the claim. $\square$

Chapter 3

Krylov–Safonov-Type Estimates

In this chapter we obtain a nonlocal version of Aleksandrov–Bakelman–Pucci estimates, Harnack inequality, and Hölder estimates for viscosity solutions to fully nonlinear nonlocal operators with kernels of variable orders. Since the early 2000s, the Harnack inequalities and Hölder estimates for nonlocal operators have been studied extensively. We refer the reader to [7, 79, 5, 6] for probabilisitic approaches. The first purely analytic method—in the spirit of the Krylov–Safonov theory—was provided by Silvestre in [76]. However, these results do not recover the classical results for second order differential equations as limit. The first robust estimates that do not blow up as the order of the equation approaches 2 was given by Caffarelli and Silvestre [15]. Their results make the theories of ingegro-differential equations and second order differential equations unified. They considered fully nonlinear integro-differential operators with kernels comparable to those of the fractional Laplacian. Namely, they considered the class $\mathcal{L}_0(\sigma)$ of linear operators of the form (1.2.4) with measurable kernels K satisfying

$$\lambda\frac{2-\sigma}{|y|^{n+\sigma}} \leq K(y) \leq \Lambda\frac{2-\sigma}{|y|^{n+\sigma}}, \tag{3.0.1}$$

where $0 < \sigma < 2$ is a fixed constant. More generally, in [53], S. Kim, Y.-C. Kim, and K.-A. Lee generalized these results to fully nonlinear integro-differential operators with regularly varying kernels. The class they considered in [53] consists of linear operators of the form (1.2.4) with measurable kernels K satisfying

$$\lambda(2-\sigma)\frac{l(|y|)}{|y|^n} \leq K(y) \leq \Lambda(2-\sigma)\frac{l(|y|)}{|y|^n},$$

where $l : (0,\infty) \to (0,\infty)$ is a locally bounded, regularly varying function at zero with an index $-\sigma$. See [53, Property 1.1 and Property 1.2] for the detailed assumptions.

In both results [15, 53], the constant $2-\sigma$ plays a very important role in the robust estimates. They used the constant $2-\sigma$ instead of the constant $C(n,\sigma)$ for the fractional Laplacian because they focused on regularity estimates which remain uniform as σ approaches 2, and two constants have the same asymptotic behavior as σ approaches 2.

In this chapter, we consider fully nonlinear integro-differential operators with kernels of variable orders. As we have already discussed in Section 1.2, the generalized constant $C(n,\varphi)$ will play a fundamental role in the regularity estimates that are robust. Let us recall the class $\mathcal{L}_0(\varphi)$ of linear operators of the form (1.2.4) with measurable kernels K satisfying (1.2.6), where φ is a function satisfying the weak scaling condition (2.1.1) with constants $0 < \sigma_1 \leq \sigma_2 < 2$ and $a \geq 1$.

Throughout this chapter, let us write $C_\varphi = C(n,\varphi)$ for the sake of brevity. Let us recall that the scale functions $\underline{C}_\varphi$ and $\overline{C}_\varphi$ are defined by (2.1.2). We introduce a new scale function

$$\Phi(R) = \frac{\underline{C}_\varphi(1) + \overline{C}_\varphi(1)}{\underline{C}_\varphi(R)}R^2,$$

which corresponds $\frac{2}{\sigma}R^\sigma$ for the case of the fractional Laplacian. This scale function has been used recently (see [1] and references therein). It will be used in order to track down viscosity solutions in every scale to find scale invariant uniform estimates because our equations are not scale invariant.

For a function u that is not positive outside the ball B_R we consider the *concave envelope* Γ *of* u^+ *in* B_{3R}, which is defined by

$$\Gamma(x) := \begin{cases} \min\{p(x) : p \text{ is a plane such that } p \geq u^+ \text{ in } B_{3R}\} & \text{in } B_{3R}, \\ 0 & \text{in } \mathbb{R}^n \setminus B_{3R}. \end{cases}$$

Let us denote by $\mathcal{C}$ the *contact set* $\{u = \Gamma\} \cap \overline{B_R}$. We are now ready to state the main theorems of this chapter. The first theorem is a nonlocal version of Aleksandrov–Bakelman–Pucci estimates, which is referred as ABP estimate for short. Let $r_0 = \rho_0 2^{-1/(2-\sigma_1)} R$ with $\rho_0 = 2^{-8} n^{-1}$.

Theorem 3.0.1 (ABP estimate)**.** *Let $u \leq 0$ in $\mathbb{R}^n \setminus B_R$ and let Γ be the concave envelope of u_+ in B_{3R}. Assume that $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u \geq -f$ in B_R in the viscosity sense. There is a universal constant $C > 0$, depending only on n, λ, and a, such that*

$$\sup_{B_R} u_+ \leq C \frac{\Phi(R)}{R} \left(\sum_j \left(\sup_{\overline{\mathcal{Q}}_j} f_+ \right)^n |\mathcal{Q}_j| \right)^{1/n}, \tag{3.0.2}$$

where $\{\mathcal{Q}_j\}$ is a finite family of pairwise disjoint open cubes, with diameters $d_j \leq r_0$, satisfying $\overline{\mathcal{Q}}_j \cap \mathcal{C} \neq \emptyset$ for each j and $\mathcal{C} \subset \bigcup_j \overline{\mathcal{Q}}_j$.

We remark that $r_0 = \rho_0 2^{-1/(2-\sigma_1)} R$ is chosen so that $r_0 \to 0$ and that the sums in the right-hand side of (3.0.2) converges to $\|f_+\|_{L^n(\mathcal{C})}$ as $\sigma_1 \to 2$. Since the constant C does not depend on σ_1 and σ_2 , Theorem 3.0.1 recovers the classical ABP estimates for second order differential equations as the limit of the Riemann sums.

Using the ABP estimates, we obtain the Harnack inequality and Hölder estimates for viscosity solutions.

Theorem 3.0.2 (Harnack inequality)**.** *Let $\sigma_0 \in (0, 2)$ and assume $\sigma_1 \geq \sigma_0$.*

Let u be a nonnegative function in $\mathbb{R}^n$ satisfying

$$\mathcal{M}^{-}_{\mathcal{L}_0(\varphi)}u \le C_0 \quad \textit{and} \quad \mathcal{M}^{+}_{\mathcal{L}_0(\varphi)}u \ge -C_0 \quad \textit{in } B_{2R}$$

in the viscosity sense. Then

$$\sup_{B_R} u \le C\left(\inf_{B_R} u + C_0\Phi(R)\right), \tag{3.0.3}$$

for some universal constant $C > 0$, depending only on n, λ, Λ, a, and σ_0.

Theorem 3.0.3 (Hölder regularity)**.** *Let $\sigma_0 \in (0,2)$ and assume $\sigma_1 \ge \sigma_0$. Let u be a bounded function in $\mathbb{R}^n$ satisfying*

$$\mathcal{M}^{-}_{\mathcal{L}_0(\varphi)}u \le C_0 \quad \textit{and} \quad \mathcal{M}^{+}_{\mathcal{L}_0(\varphi)}u \ge -C_0 \quad \textit{in } B_{2R}$$

in the viscosity sense. Then $u \in C^{\alpha}(\overline{B_R})$ and

$$\|u\|'_{C^{\alpha}(\overline{B_R})} \le C\left(\|u\|_{L^{\infty}(\mathbb{R}^n)} + C_0\Phi(R)\right) \tag{3.0.4}$$

for some universal constants $\alpha \in (0,1)$ and $C > 0$, depending only on n, λ, Λ, a, and σ_0.

For the $C^{1,\alpha}$ regularity result, we introduce a class $\mathcal{L}_1(\varphi)$ of linear operators in $\mathcal{L}_0(\varphi)$ with more regular kernels K satisfying

$$\int_{\mathbb{R}^n\setminus B_\rho} \frac{|K(y)-K(y-h)|}{|h|}\,\mathrm{d}y \le C \quad \text{every time } |h| < \rho/2, \tag{3.0.5}$$

for given $\rho > 0$.

Theorem 3.0.4. *Assume $\sigma_1 \ge \sigma_0 > 0$ and let $\mathcal{I}$ be a translation invariant elliptic operator with respect to $\mathcal{L}_1(\varphi)$. If u is a bounded function satisfying $\mathcal{I}u = 0$ in B_{2R} in the viscosity sense, then $u \in C^{1,\alpha}(\overline{B_R})$ and*

$$\|u\|'_{C^{1,\alpha}(\overline{B_R})} \le C\left(\|u\|_{L^{\infty}(\mathbb{R}^n)} + \Phi(R)\|\mathcal{I}0\|_{L^{\infty}(\mathbb{R}^n)}\right) \tag{3.0.6}$$

for some universal constants $\alpha \in (0,1)$ and $C > 0$, depending only on n, λ, Λ, a, σ_0, and the constant in (3.0.5).

We remark that higher order regularity results, such as $C^{\sigma+\alpha}$ regularity estimates, will be established in the next chapter even for more general operators.

It is important to note that in the regularity estimates (3.0.3), (3.0.4), and (3.0.6), the constants are independent of σ_1 and σ_2, but the term $\Phi(R)$ in the right-hand side of (3.0.3), (3.0.4), and (3.0.6) still depends on σ_1 and σ_2. For the fractional Laplacian case this term corresponds to $\frac{2}{\sigma}R^\sigma$ and it can be further estimated as

$$\frac{2}{\sigma}R^\sigma \leq \frac{2}{\sigma_0}R^\sigma.$$

In our case, we can also estimate the term $\Phi(R)$ using Lemma 2.1.3 and Lemma 2.1.1 as

$$\begin{aligned} \Phi(R) &\leq \frac{c_2}{c_1}\frac{(\underline{C}_\varphi(R) + \overline{C}_\varphi(R))R^2}{\underline{C}_\varphi(R)} \\ &\leq \frac{c_2}{c_1}\left(R^2 + \frac{2a^2}{\sigma_0}\right) \leq C(n,a,\sigma_0)(R^2+1), \end{aligned} \tag{3.0.7}$$

which is independent of σ_1 and σ_2. Nevertheless, we leave (3.0.3), (3.0.4), and (3.0.6) as they are because the estimate (3.0.7) has a different scale with respect to R. The results in this chapter are based on the joint work in [50].

This chapter is organized as follows. In Section 3.1 we focus on the discrete version of the ABP estimates, from which Theorem 3.0.1 will follow. Section 3.2 is devoted to the construction of a barrier function. This barrier function is utilized, together with the ABP estimates, in order to prove the power decay estimates of sub-level sets of the viscosity supersolutions in Section 3.3. The power decay estimates will provide the weak Harnack inequality, local boundedness, and Harnack inequalitiy in Section 3.4. In Section 3.5, the Hölder estimates and $C^{1,\alpha}$ estimates are established.

3.1 Aleksandrov–Bakelman–Pucci Estimates

The first step towards to the regularity results is the ABP estimate which is the fundamental tool in the regularity theory for fully nonlinear operators. This section is devoted to the proof of Theorem 3.0.1.

Let us first prove that there is at least one good ring near a contact point where u stays quadratically close the tangent plane to Γ at the contact point. We set $r_k = \rho_0 2^{-1/(2-\sigma_1)-k} R$ in the following lemmas.

Lemma 3.1.1. *Let u and Γ be functions as in Theorem 3.0.1. There is a constant $C > 0$, depending only on n, λ, and a, such that for each $x \in \mathcal{C}$ and $M > 0$, there is an integer $k \geq 0$ satisfying*

$$|A_k| \leq C \frac{f(x)}{M} |R_k|, \tag{3.1.1}$$

where $R_k = B_{r_k}(x) \setminus B_{r_{k+1}}(x)$,

$$A_k = \left\{ y \in R_k(x) : u(y) < u(x) + (y - x) \cdot \nabla\Gamma(x) - M \frac{\Phi(r_0)}{r_0^2} r_k^2 \right\},$$

and $\nabla\Gamma$ stands for an element of the super-differential of Γ at x.

Proof. Let $x \in \mathcal{C}$ be a point such that $u(x) = \Gamma(x) > 0$. By Lemma 2.3.2, $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u(x)$ is defined in the classical sense and $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u(x) \geq -f(x)$. Since $\delta(u, x, y) \leq 0$ for all $y \in \mathbb{R}^n$, we have

$$-f(x) \leq \mathcal{M}^+_{\mathcal{L}_0(\varphi)} u(x) = C_\varphi \int_{\mathbb{R}^n} \frac{-\lambda \delta_-(u, x, y)}{|y|^n \varphi(|y|)} \, \mathrm{d}y. \tag{3.1.2}$$

Indeed, if both $x + y$ and $x - y$ are in B_{3R}, then $\delta(u, x, y) \leq 0$ since Γ is concave and lies above u, and otherwise both $x + y$ and $x - y$ are not in B_R, which implies $u(x + y) \leq 0$ and $u(x - y) \leq 0$.

We split the integral in the right-hand side of (3.1.2) as

$$f(x) \geq \lambda C_\varphi \sum_{k=0}^\infty \int_{A_k - x} \frac{\delta_-(u, x, y)}{|y|^n \varphi(|y|)} \, \mathrm{d}y.$$

If $y \in A_k - x$, then we have $x + y \in B_{3R}$ and $x - y \in B_{3R}$, yielding that

$$\begin{aligned}\delta(u, x, y) &< \left(u(x) + y \cdot \nabla\Gamma(x) - M\frac{\Phi(r_0)}{r_0^2} r_k^2 \right) + (\Gamma(x) - y \cdot \nabla\Gamma(x)) - 2u(x) \\ &= -M\frac{\Phi(r_0)}{r_0^2} r_k^2.\end{aligned}$$

We thus have by means of the weak scaling condition (2.1.1) that

$$f(x) \geq \frac{\lambda}{a} C_\varphi \frac{\Phi(r_0)}{r_0^2} \sum_{k=0}^{\infty} \frac{M r_k^2}{r_k^n \varphi(r_k)} |A_k|.$$

Suppose that we cannot find an integer $k \geq 0$ satisfying (3.1.1) with some constant $C > 0$ which will be chosen later. Then we have

$$f(x) > \frac{\lambda}{a} C_\varphi \frac{\Phi(r_0)}{r_0^2} \sum_{k=0}^{\infty} \frac{M r_k^2}{r_k^n \varphi(r_k)} C \frac{f(x)}{M} |R_k| = C \frac{3\omega_n \lambda}{4a} C_\varphi \frac{\Phi(r_0)}{r_0^2} \sum_{k=0}^{\infty} \frac{r_k^2}{\varphi(r_k)} f(x).$$

The weak scaling condition (2.1.1) shows that

$$\begin{aligned}\underline{C}_\varphi(r_0) &= \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \frac{r}{\varphi(r)} \, \mathrm{d}r \leq \sum_{k=0}^{\infty} \int_{r_{k+1}}^{r_k} \frac{a r_k}{\varphi(r_k)} \left(\frac{r_k}{r} \right)^{\sigma_2} \mathrm{d}r \\ &\leq \sum_{k=0}^{\infty} \frac{a r_k (r_k - r_{k+1})}{\varphi(r_k)} \left(\frac{r_k}{r_{k+1}} \right)^2 = 4a \sum_{k=0}^{\infty} \frac{r_k^2}{\varphi(r_k)}.\end{aligned} \tag{3.1.3}$$

Therefore, by the definition of Φ, Lemma 2.1.3, and (3.1.3), we arrive at

$$f(x) > C \frac{3\omega_n \lambda}{16a^2} c_1 f(x),$$

which yields a contradiction if we take C large so that $C \geq 16a^2/(3\omega_n \lambda c_1)$. $\square$

We remark that (3.1.2) implies that $f(x) > 0$ for a contact point $x \in \mathcal{C}$.

Lemma 3.1.2. *Let u and Γ be functions as in Theorem 3.0.1. There are constants $\varepsilon = \varepsilon_n \in (0, 1)$ and $C = C(n, \lambda, a) > 0$ such that for each $x \in \mathcal{C}$,* *there is an integer $k \geq 0$ satisfying*

$$\left|\left\{y \in R_k : u(y) < u(x) + (y - x) \cdot \nabla\Gamma(x) - C\frac{\Phi(r_0)}{r_0^2} f(x) r_k^2\right\}\right| \leq \varepsilon_n |R_k| \tag{3.1.4}$$

and

$$|\nabla\Gamma(B_{r_k/4}(x))| \leq C \left(\frac{\Phi(r_0)}{r_0^2} f(x)\right)^n |B_{r_k/4}(x)|. \tag{3.1.5}$$

Proof. Let $\varepsilon = \varepsilon_n$ be the constant in [15, Lemma 8.4], then the inequality (3.1.4) follows from Lemma 3.1.1 by choosing $M = Cf(x)/\varepsilon$. For (3.1.5), we use [15, lemma 8.4] and the concavity of Γ. $\square$

The following theorem is a discrete version of ABP estimates, which will produce Theorem 3.0.1.

Theorem 3.1.3 (Discrete ABP estimate)**.** *Let u and Γ be functions as in Theorem 3.0.1. There is a finite family $\{\mathcal{Q}_j\}$ of pairwise disjoint open cubes with diameters $d_j \leq r_0$ such that the following holds:*

(i) $\mathcal{C} \cap \overline{\mathcal{Q}}_j \neq \emptyset$ *for any* $\mathcal{Q}_j$.

(ii) $\mathcal{C} \subset \bigcup_j \overline{\mathcal{Q}}_j$.

(iii) $|\nabla\Gamma(\overline{\mathcal{Q}}_j)| \leq C \left(\Phi(r_0) r_0^{-2} \sup_{\overline{\mathcal{Q}}_j} f_+\right)^n |\mathcal{Q}_j|$.

(iv) $\left|\left\{y \in 32\sqrt{n}\mathcal{Q}_j : u(y) \geq \Gamma(y) - C\Phi(r_0) \sup_{\overline{\mathcal{Q}}_j} f_+\right\}\right| \geq \mu |\mathcal{Q}_j|$.

The constants $C > 0$ and $\mu > 0$ depend only on n, λ, and a.

The proof of Theorem 3.1.3 is the same with the proof of [15, Theorem 8.7]. To prove Theorem 3.0.1 we use the following lemma.

Lemma 3.1.4. *If $r_0 = \rho_0 2^{-1/(2-\sigma_1)} R$, then*

$$\left(\frac{R}{r_0}\right)^2 \left(\frac{\Phi(R)}{\Phi(r_0)}\right)^{-1} \leq C \tag{3.1.6}$$

for some constant $C = C(a, \rho_0)$.

Proof. By the definition of Φ, we have

$$\left(\frac{R}{r_0}\right)^2 \left(\frac{\Phi(R)}{\Phi(r_0)}\right)^{-1} = \frac{\underline{C}_\varphi(R)}{\underline{C}_\varphi(r_0)}.$$

By Lemma 2.1.2, we obatin

$$\frac{\underline{C}_\varphi(R)}{\underline{C}_\varphi(r_0)} \leq 1 + a^2 \left(\rho_0 2^{-1/(2-\sigma_1)}\right)^{-(2-\sigma_1)} \leq 1 + 2a^2\rho_0^2.$$

Thus, (3.1.6) holds with $C = 1 + 2a^2\rho_0^{-2}$. □

We are now ready to prove Theorem 3.0.1.

Proof of Theorem 3.0.1. Let $\{\mathcal{Q}_j\}$ be the finite family of cubes constructed in Theorem 3.1.3. Since

$$\left\{p \in \mathbb{R}^n : |p| \leq \frac{\sup_{B_R} u_+}{4R}\right\} \subset \nabla\Gamma(B_{3R}) = \nabla\Gamma(\mathcal{C}),$$

we have in light of Theorem 3.1.3 (ii) that

$$\left(\frac{\sup_{B_R} u_+}{4R}\right)^n \leq C|\nabla\Gamma(\mathcal{C})| \leq C\sum_j |\nabla\Gamma(\overline{\mathcal{Q}}_j)|.$$

Moreover, by Theorem 3.1.3 (iii), we obtain

$$\sup_{B_R} u_+ \leq CR\frac{\Phi(r_0)}{r_0^2}\left(\sum_j \left(\sup_{\overline{\mathcal{Q}}_j} f\right)_+^n |\mathcal{Q}_j|\right)^{1/n}. \tag{3.1.7}$$

Therefore, (3.1.7) finishes the proof with the help of (3.1.6). □

3.2 A Barrier Function

This section is devoted to a construction of a barrier function at every scale in order to find scale invariant uniform estimates. Recall that ρ_0 was defined by $\rho_0 = 2^{-8}n^{-1}$, but the following lemma holds for any $\rho_0 \in (0, 1)$.

Lemma 3.2.1. *Assume $\sigma_1 \geq \sigma_0 > 0$. There are universal constants $p > n+1$ and $\kappa \in (0, \rho_0/8)$, depending only on n, λ, Λ, a, and σ_0, such that the function $v_1(x) = \min\{|\kappa R|^{-p}, |x|^{-p}\}$ satisfies $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v_1(x) \geq 0$ for all $x \in B_R \setminus B_{\rho_0 R}$.*

Proof. Without loss of generality we may assume that $x = R_0 e_1$ for $\rho_0 R \leq R_0 < R$. We need to compute

$$\begin{aligned}
&\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v_1(x) \\
&= C_\varphi \left(\int_{B_{R_0/2}} \frac{\frac{\lambda}{2}\delta_+ - \Lambda\delta_-}{|y|^n \varphi(|y|)} \,\mathrm{d}y + \int_{B^c_{R_0/2}} \frac{\frac{\lambda}{2}\delta_+ - \Lambda\delta_-}{|y|^n \varphi(|y|)} \,\mathrm{d}y + \int_{\mathbb{R}^n} \frac{\frac{\lambda}{2}\delta_+}{|y|^n \varphi(|y|)} \,\mathrm{d}y \right) \\
&=: C_\varphi (I_1 + I_2 + I_3),
\end{aligned}$$

where $\delta = \delta(v_1, x, y)$.

For $|y| \leq R_0/2$, we have

$$\begin{aligned}
\delta(v_1, x, y) &= R_0^{-p} \left(\left| \frac{x}{R_0} + \frac{y}{R_0} \right|^{-p} + \left| \frac{x}{R_0} - \frac{y}{R_0} \right|^{-p} - 2 \right) \\
&\geq \frac{p}{R_0^p} \left(- \left| \frac{y}{R_0} \right|^2 + (p+2)\frac{y_1^2}{R_0^2} - \frac{1}{2}(p+2)(p+4)\frac{y_1^2 |y|^2}{R_0^4} \right).
\end{aligned}$$

We choose $p = p(n, \lambda, \Lambda) > n+1$ sufficiently large so that

$$\frac{\lambda}{2}(p+2) \int_{\partial B_1} y_1^2 \,\mathrm{d}\sigma(y) - \Lambda |\partial B_1| \geq 0.$$

Then we obtain

$$\begin{aligned}
I_1 &\geq \frac{p}{R_0^p} \int_{B_{R_0/2}} \left(\frac{\lambda}{2}(p+2)\frac{y_1^2}{R_0^2} - \Lambda \frac{|y|^2}{R_0^2} - \Lambda \frac{(p+2)(p+4) y_1^2 |y|^2}{2R_0^4} \right) \frac{\mathrm{d}y}{|y|^n \varphi(|y|)} \\
&\geq -\frac{p}{R_0^p} \frac{\Lambda(p+2)(p+4)}{2R_0^4} c_n \int_0^{R_0/2} \frac{r^3}{\varphi(r)} \,\mathrm{d}r,
\end{aligned}$$

where $c_n = \int_{\partial B_1} y_1^2 \,\mathrm{d}\sigma(y) > 0$ is a constant depending only on n. Using the weak scaling condition (2.1.1), we have

$$\int_0^{R_0/2} \frac{r^3}{\varphi(r)}\,\mathrm{d}r \le \frac{a}{4-\sigma_2}\frac{R_0^4}{16\varphi(R_0/2)} \le \frac{aR_0^4}{32\varphi(R_0/2)},$$

and hence

$$I_1 \ge -\frac{\Lambda p(p+2)(p+4)c_n a}{64R_0^p\varphi(R_0/2)}. \tag{3.2.1}$$

On the other hand, using the inequality (2.1.4) we estimate

$$\begin{aligned} I_2 &\ge -\frac{1}{R_0^p}\int_{\mathbb{R}^n\setminus B_{R_0/2}} \frac{2\Lambda}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\ &= -\frac{2n\omega_n\Lambda}{R_0^p}\overline{C}_\varphi(R_0/2) \ge -\frac{2n\omega_n\Lambda}{a\sigma_1 R_0^p\varphi(R_0/2)}. \end{aligned} \tag{3.2.2}$$

Finally, let us make I_3 sufficiently large by selecting $\kappa > 0$ small. We have

$$\begin{aligned} I_3 &\ge \frac{\lambda}{2}\int_{B_{R_0/4}(x)} \frac{|x-y|^{-p} - 2R_0^{-p}}{|y|^n\varphi(|y|)}\,\mathrm{d}y \ge \frac{\lambda}{4}\int_{B_{R_0/4}(x)\setminus B_{\kappa R}(x)} \frac{|x-y|^{-p}}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\ &= \frac{\lambda}{4}\int_{B_{R_0/4}\setminus B_{\kappa R}} \frac{|z|^{-p}}{|x+z|^n\varphi(|x+z|)}\,\mathrm{d}z \ge \frac{\lambda n\omega_n m}{2^{n+2}R_0^n}\int_{\kappa R}^{R_0/4} r^{-p+n-1}\,\mathrm{d}r, \end{aligned}$$

where $m = \min_{r\in[R_0/2,3R_0/2]} 1/\varphi(r)$. If we have taken $\kappa \in (0, \rho_0/8)$, then we obtain

$$\begin{aligned} \int_{\kappa R}^{R_0/4} r^{-p+n-1}\,\mathrm{d}r &= \frac{(\kappa R)^{-p+n} - (R_0/4)^{-p+n}}{p-n} \\ &\ge \frac{(\rho_0/\kappa)^{p-n} - 4^{p-n}}{p-n}R_0^{n-p} \ge \frac{1}{2(p-n)}\frac{\rho_0}{\kappa}R_0^{n-p}. \end{aligned}$$

We use the weak scaling condition (2.1.1) to estimate

$$m \ge \frac{1}{a\varphi(3R_0/2)} \ge \frac{1}{3^{\sigma_2}a^2\varphi(R_0/2)} \ge \frac{1}{3^2a^2\varphi(R_0/2)},$$

and hence

$$I_3 \ge \frac{\lambda n\omega_n}{3^2 2^{n+3}a^2(p-n)}\frac{\rho_0}{\kappa}\frac{1}{R_0^p\varphi(R_0/2)}. \tag{3.2.3}$$

Combining (3.2.1)–(3.2.3), we have

$$I_1+I_2+I_3 \geq \left(\frac{\lambda n \omega_n \rho_0/\kappa}{3^2 2^{n+3} a^2 (p-n)} - \frac{\Lambda p(p+2)(p+4) c_n a}{64} - \frac{2\Lambda n \omega_n}{a\sigma_0} \right) \frac{R_0^{-p}}{\varphi(R_0/2)}.$$

By taking $\kappa = \kappa(n, \lambda, \Lambda, a, \sigma_0, \rho_0) \in (0, \rho_0/8)$ sufficiently small, we have $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v_1(x) = C_\varphi (I_1+I_2+I_3) \geq 0$ for all $x \in B_R \setminus B_{\rho_0 R}$. □

Corollary 3.2.2. *Assume $\sigma_1 \geq \sigma_0 > 0$. There is a continuous function v such that $v = 0$ outside B_R, $v > 2$ in $B_{3R/4}$, and $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v \geq -\psi$ in $\mathbb{R}^n$ for some nonnegative bounded function ψ supported in $\overline{B_{\rho_0 R}}$.*

Proof. Let v_1 be the function in Lemma 3.2.1 and define

$$v(x) := c_0 \begin{cases} P(x) & \text{for } x \in B_{\kappa R}, \\ (\kappa R)^p (v_1(x) - v_1(R)) & \text{for } x \in B_R \setminus B_{\kappa R}, \\ 0 & \text{for } x \in B_R^c, \end{cases}$$

where $c_0 := \frac{2}{\kappa^p((4/3)^p - 1)}$ and $P(x) := -a|x|^2 + b$ with $a = \frac{1}{2} p (\kappa R)^{-2}$ and $b = 1 - \kappa^p + \frac{1}{2}p$. Then v is continuous in $\mathbb{R}^n$, $C^{1,1}$ in B_R, and $v \geq 2$ in $B_{3R/4}$. Thus, it remains to show that $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v \geq -\psi$ in $\mathbb{R}^n$. Indeed, if $x \in B_R \setminus B_{\rho_0 R}$, then since $\delta(v,x,y) \geq c_0 (\kappa R)^p \delta(v_1, x, y)$, we have by Lemma 3.2.1,

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v(x) \geq c_0 (\kappa R)^p \mathcal{M}^-_{\mathcal{L}_0(\varphi)} v_1(x) \geq 0.$$

If $x \in \mathbb{R}^n \setminus B_R$, then we have $\delta(v,x,y) \geq 0$ and hence $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v(x) \geq 0$. Finally, if $x \in B_{\rho_0 R}$, then by using Lemma 2.1.3 we obtain

$$\begin{aligned} \Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v(x) &\geq -\Lambda C_\varphi \Phi(R) \int_{\mathbb{R}^n} \frac{\delta_-(v,x,y)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\ &\geq -\Lambda \frac{c_2}{\underline{C}_\varphi(R)} R^2 \left(\int_{B_R} \frac{cR^{-2}|y|^2}{|y|^n \varphi(|y|)} \,\mathrm{d}y + \int_{B_R^c} \frac{2bc_0}{|y|^n \varphi(|y|)} \,\mathrm{d}y \right) \\ &\geq -C - C \frac{\overline{C}_\varphi(R)}{\underline{C}_\varphi(R)} R^2, \end{aligned}$$

where we have used $D^2 v \geq -cR^{-2}I$ a.e. in B_R for some constant $c > 0$. Note that Lemma 2.1.1 shows that

$$\frac{\overline{C}_\varphi(R)}{\underline{C}_\varphi(R)} R^2 \leq \frac{a^2(2-\sigma_1)}{\sigma_1} \leq \frac{2a^2}{\sigma_0}. \tag{3.2.4}$$

Therefore, we conclude that $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} v \geq -\psi$ in $\mathbb{R}^n$ for some nonnegative function satisfying $\operatorname{supp}\psi \subset B_{\rho_0 R}$ and a uniform bound $\|\psi\|_\infty \leq C$. □

3.3 Power Decay Estimates

In this section, we establish the measure estimates of sub-level sets of the viscosity supersolutions to fully nonlinear nonlocal elliptic equations with respect to $\mathcal{L}_0(\varphi)$, using the ABP estimates and the barrier function constructed in Section 3.2. The following is a key lemma for power decay estimates. Let us recall that $Q_R = Q_R(0)$ denotes a cube with center 0 and side R.

Lemma 3.3.1. *Assume $\sigma_1 \geq \sigma_0 > 0$. There exist universal constants $\varepsilon_0, \mu_0 \in (0,1)$, and $M_0 > 1$, depending only on n, λ, Λ, a, and σ_0, such that if a nonnegative function u satisfies $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq \varepsilon_0$ in Q_{2R} in the viscosity sense and $\inf_{Q_{3R/(2\sqrt{n})}} u \leq 1$, then*

$$\left|\{u \leq M_0\} \cap Q_{R/(2\sqrt{n})}\right| > \mu_0 \left|Q_{R/(2\sqrt{n})}\right|.$$

Proof. Let v be the barrier constructed in Corollary 3.2.2 and let us consider the function $w := v - u$. It satisfies that $w \leq 0$ outside B_R, $\sup_{B_R} w \geq 1$, and

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)} w \geq \mathcal{M}^-_{\mathcal{L}_0(\varphi)} v - \mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \geq -\frac{1}{\Phi(R)}(\psi + \varepsilon_0) \quad \text{in } B_R$$

in the viscosity sense. Let Γ_w be the concave envelope of w_+ in B_{3R}, then Theorem 3.0.1 shows that

$$1 \leq \sup_{B_R} w_+ \leq C\frac{\Phi(R)}{R}\left(\sum_j \left\|\frac{\psi + \varepsilon_0}{\Phi(R)}\right\|^n_{L^\infty(\mathcal{Q}_j)} |\mathcal{Q}_j|\right)^{1/n},$$

where $\{\mathcal{Q}_j\}$ is a finite family of pairwise disjoint open cubes constructed in Theorem 3.1.3 for w and Γ_w.

Since $\operatorname{supp}\psi \subset \overline{B_{\rho_0 R}}$, $\|\psi\|_\infty \le C$, and $\sum_j |\mathcal{Q}_j| \le CR^n$, we have

$$1 \le \frac{C}{R}\left(\sum_{\overline{\mathcal{Q}}_j \cap \overline{B_{\rho_0 R}} \neq \emptyset} |\mathcal{Q}_j| \right)^{1/n} + C\varepsilon_0.$$

By taking $\varepsilon_0 > 0$ sufficiently small, we obtain that

$$\left|Q_{R/(2\sqrt{n})}\right| \le C \sum_{\overline{\mathcal{Q}}_j \cap \overline{B_{\rho_0 R}} \neq \emptyset} |\mathcal{Q}_j|,$$

for some universal constant $C > 0$. In light of Theorem 3.1.3 (iv), we have

$$\left| \left\{ y \in 32\sqrt{n}\mathcal{Q}_j : w(y) \ge \Gamma_w(y) - C\Phi(r_0)\frac{\|\psi + \varepsilon_0\|_\infty}{\Phi(R)} \right\} \right| \ge \mu|\mathcal{Q}_j|$$

for all j. Since $r_0 < R$, it follows from Lemma 3.1.4 that $\frac{\Phi(r_0)}{\Phi(R)} \le C(r_0/R)^2 \le C$. Recalling that $w = v - u$ and that $\|\psi + \varepsilon_0\|_\infty \le C$, we obatin that

$$\left|\left\{ y \in 32\sqrt{n}\mathcal{Q}_j : u(y) \le M_0 \right\}\right| \ge \mu|\mathcal{Q}_j|$$

for some universal constant $M_0 > 1$. From the choice of $\rho_0 = 2^{-8}n^{-1}$, we know that $32\sqrt{n}\mathcal{Q}_j \subset B_{R/(4\sqrt{n})} \subset Q_{R/(2\sqrt{n})}$ for any $\mathcal{Q}_j$ satisfying $\overline{\mathcal{Q}}_j \cap \overline{B_{\rho_0 R}} \neq \emptyset$. Taking a subfamily of $\{32\sqrt{n}\mathcal{Q}_j : \overline{\mathcal{Q}}_j \cap \overline{B_{\rho_0 R}} \neq \emptyset\}$ with finite overlapping, we obtain that

$$\left|Q_{R/(2\sqrt{n})}\right| \le C \sum_{\overline{\mathcal{Q}}_j \cap \overline{B_{\rho_0 R}} \neq \emptyset} |\mathcal{Q}_j| \le C \left| \{u \le M_0\} \cap Q_{R/(2\sqrt{n})} \right|.$$

Therefore, we conclude the lemma by taking $\mu_0 = 1/C$. □

It is now standard to obtain the following lemma as a consequence of Lemma 3.3.1. This is analogous to [18, Lemma 4.6] and [15, Lemma 10.2].

Lemma 3.3.2. *Assume $\sigma_1 \ge \sigma_0 > 0$. Let ε_0, μ_0, M_0, and u be as in*

Lemma 3.3.1. Then,

$$\left|\left\{u > M_0^k\right\} \cap Q_{R/(2\sqrt{n})}\right| \leq (1-\mu_0)^k \left|Q_{R/(2\sqrt{n})}\right|$$

for all $k \in \mathbb{N}$. As a consequence, we have that

$$\left|\{u > t\} \cap Q_{R/(2\sqrt{n})}\right| \leq CR^n t^{-\varepsilon}$$

for all $t > 0$, where $C > 0$ and $\varepsilon > 0$ are universal constants depending only on n, λ, Λ, a, and σ_0.

By the standard covering argument, we have the following theorem which is referred in the literature as L^ε-estimate (see [18]).

Theorem 3.3.3. *Assume $\sigma_1 \geq \sigma_0 > 0$. Let u be a nonnegative function in $\mathbb{R}^n$ such that $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq C_0$ in B_{2R} in the viscosity sense. Then*

$$|\{u > t\} \cap B_R| \leq CR^n \left(u(0) + C_0 \Phi(R)\right)^\varepsilon t^{-\varepsilon}$$

for all $t > 0$, where $C > 0$ and $\varepsilon > 0$ are universal constants depending only on n, λ, Λ, a, and σ_0.

3.4 The Harnack Inequality

This section is devoted to the proof of Theorem 3.0.2. Let us first state the weak Harnack inequality, which provides a half of the Harnack inequality.

Theorem 3.4.1 (Weak Harnack inequality)**.** *Assume $\sigma_1 \geq \sigma_0 > 0$. Let u be a nonnegative function in $\mathbb{R}^n$ such that $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq C_0$ in B_{2R} in the viscosity sense. Then*

$$\left(\fint_{B_R} |u|^p\right)^{1/p} \leq C\left(u(0) + C_0 \Phi(R)\right)$$

for some universal constants $C > 0$ and $p > 0$, depending only on n, λ, Λ, a, and σ_0.

Note that Theorem 3.4.1 is immediate from Theorem 3.3.3 (see [18, Theorem 4.8] for the proof).

Let us next prove the following version of the Harnack inequality, from which Theorem 3.0.2 follows by the standard covering argument.

Theorem 3.4.2 (Harnack inequality)**.** *Assume $\sigma_1 \geq \sigma_0 > 0$. Let u be a nonnegative function in $\mathbb{R}^n$ such that*

$$\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq C_0 \quad \text{and} \quad \Phi(R)\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u \geq -C_0 \quad \text{in } B_{2R}$$

in the viscosity sense. Then

$$\sup_{B_{R/2}} u \leq C\left(u(0) + C_0\right)$$

for some universal constant $C > 0$, depending only on n, λ, Λ, a, and σ_0.

Proof. We may assume that $u > 0$ and $u(0) + C_0 \leq 1$. Let $\varepsilon > 0$ be the constant as in Theorem 3.3.3 and let $\gamma = (n+2)/\varepsilon$. Let us consider the minimal value of $t > 0$ such that

$$u(x) \leq h_t(x) := t\left(1 - \frac{|x|}{R}\right)^{-\gamma} \quad \text{for all } x \in B_R, \tag{3.4.1}$$

so that there exists a point $x_0 \in B_R$ satisfying $u(x_0) = h_t(x_0)$. If we show that $t \leq C$ for some universal constant $C > 0$, it follows from (3.4.1) that $\sup_{B_{R/2}} u \leq C2^\gamma$, which finishes the proof.

Let $d = R - |x_0|, r = d/2$, and let $A = \{u > u(x_0)/2\}$. Then $u(x_0) = h_t(x_0) = t(R/d)^\gamma$. By Theorem 3.3.3, we have

$$|A \cap B_R| \leq CR^n \left(\frac{u(x_0)}{2}\right)^{-\varepsilon} \leq Ct^{-\varepsilon}R^n\left(\frac{d}{R}\right)^{\gamma\varepsilon} \leq Ct^{-\varepsilon}d^n.$$

Since $B_r(x_0) \subset B_R$ and $r = d/2$, we obtain

$$|\{u > u(x_0)/2\} \cap B_r(x_0)| \leq Ct^{-\varepsilon}|B_r(x_0)|. \tag{3.4.2}$$

This implies that if t is large, the set A can cover only a small portion of $B_r(x_0)$. We will show that there is a universal constant $\theta > 0$ such that

$$|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \leq \frac{1}{2}|B_{\theta r/4}|, \tag{3.4.3}$$

provided that t is sufficiently large. However, if we can make t arbitrarily large, then (3.4.3) will contradict to (3.4.2). Therefore, we will obtain a uniform bound for t.

Let us first estimate $|\{u < u(x_0)/2\} \cap B_{\theta r}(x_0)|$ for small $\theta > 0$, which will be chosen uniformly later. For every $x \in B_{\theta r}(x_0)$, we have

$$u(x) \leq h_t(x) \leq t\left(\frac{d-\theta r}{R}\right)^{-\gamma} = \left(1-\frac{\theta}{2}\right)^{-\gamma} u(x_0).$$

Let us take the function

$$v(x) := \left(1-\frac{\theta}{2}\right)^{-\gamma} u(x_0) - u(x).$$

Then v is nonnegative in $B_{\theta r}(x_0)$. In order to apply Theorem 3.3.3 to $w := v_+$, we compute $\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w$ in $B_{\theta r}(x_0)$. For $x \in B_{\theta r/2}(x_0)$, we have

$$\begin{aligned}
\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w(x) &\leq \mathcal{M}^-_{\mathcal{L}_0(\varphi)} v(x) + \mathcal{M}^+_{\mathcal{L}_0(\varphi)} v_-(x) \\
&\leq -\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u(x) + C_\varphi \int_{\mathbb{R}^n} \frac{\Lambda v_-(x+y) + \Lambda v_-(x-y)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\
&\leq \frac{1}{\Phi(R)} + 2\Lambda C_\varphi \int_{\{v(x+y)<0\}} \frac{v_-(x+y)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\
&\leq \frac{1}{\Phi(R)} + 2\Lambda C_\varphi \int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{(u(x+y) - (1-\frac{\theta}{2})^{-\gamma} u(x_0))_+}{|y|^n \varphi(|y|)} \,\mathrm{d}y.
\end{aligned} \tag{3.4.4}$$

To estimate the last integral in (3.4.4), we introduce a function

$$g_\beta(x) := \beta\left(1 - \frac{|4x|^2}{R^2}\right)_+,$$

and consider the largest number $\beta > 0$ such that $u \geq g_\beta$. Let $x_1 \in B_{R/4}$ be a point such that $u(x_1) = g_\beta(x_1)$. This is possible because we have assumed that $u > 0$ in B_{2R}. We observe that $\beta \leq 1$ since $u(0) \leq 1$. We estimate

$$\begin{aligned}\int_{\mathbb{R}^n} \frac{\delta_-(u, x_1, y)}{|y|^n \varphi(|y|)}\,\mathrm{d}y &\leq \int_{\mathbb{R}^n} \frac{\delta_-(g_\beta, x_1, y)}{|y|^n \varphi(|y|)}\,\mathrm{d}y \\ &\leq \int_{B_R} \frac{C|y/R|^2}{|y|^n \varphi(|y|)}\,\mathrm{d}y + \int_{\mathbb{R}^n \setminus B_R} \frac{C}{|y|^n \varphi(|y|)}\,\mathrm{d}y \\ &\leq C\left(\frac{\underline{C}_\varphi(R)}{R^2} + \overline{C}_\varphi(R) \right).\end{aligned}$$

Thus, it follows from the equation $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq 1$ that

$$\begin{aligned}C_\varphi \int_{\mathbb{R}^n} \frac{\lambda \delta_+(u, x_1, y)}{|y|^n \varphi(|y|)}\,\mathrm{d}y &= \mathcal{M}^-_{\mathcal{L}_0(\varphi)} u(x_1) + C_\varphi \int_{\mathbb{R}^n} \frac{\Lambda \delta_-(u, x_1, y)}{|y|^n \varphi(|y|)}\,\mathrm{d}y \\ &\leq \frac{1}{\Phi(R)} + CC_\varphi \left(\frac{\underline{C}_\varphi(R)}{R^2} + \overline{C}_\varphi(R) \right) \leq \frac{C}{\Phi(R)},\end{aligned}$$

where we have used Lemma 2.1.3 and (3.2.4) in the last inequality. Since $u(x_1) \leq \beta \leq 1$ and $u(x-y) > 0$, we have

$$C_\varphi \int_{\mathbb{R}^n} \frac{(u(x_1+y)-2)_+}{|y|^n \varphi(|y|)}\,\mathrm{d}y \leq \frac{C}{\Phi(R)}.$$

If $u(x_0) \leq 2$, then $t = u(x_0)(d/R)^\gamma \leq 2$, which gives a uniform bound of t. Otherwise, we can estimate the last integral in (3.4.4) for $x \in B_{\theta r/2}(x_0)$ as follows:

$$\begin{aligned}&\int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{\left(u(x+y) - (1-\frac{\theta}{2})^{-\gamma} u(x_0)\right)_+}{|y|^n \varphi(|y|)}\,\mathrm{d}y \\ &\leq \int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{(u(x+y)-2)_+}{|y|^n \varphi(|y|)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{(u(x_1+x+y-x_1)-2)_+}{|x+y-x_1|^n \varphi(|x+y-x_1|)} \frac{|x+y-x_1|^n \varphi(|x+y-x_1|)}{|y|^n \varphi(|y|)}\,\mathrm{d}y.\end{aligned}$$

We see that for $x \in B_{\theta r/2}(x_0)$ and $y \in \mathbb{R}^n \setminus B_{\theta r/2}$,

$$\frac{|x+y-x_1|}{|y|} \leq \frac{|x-x_0|+|x_0|+|y|+|x_1|}{|y|} \leq C\frac{R}{\theta r}$$

and

$$\frac{\varphi(|x+y-x_1|)}{\varphi(|y|)} \leq \frac{\varphi(CR+|y|)}{\varphi(|y|)} \leq a\left(\frac{CR}{\theta r}\right)^{\sigma_2} \leq C\left(\frac{R}{\theta r}\right)^2,$$

with the help of the weak scaling condition (2.1.1). Thus, we obtain

$$\int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{(u(x+y)-(1-\frac{\theta}{2})^{-\gamma}u(x_0))_+}{|y|^n\varphi(|y|)}\,\mathrm{d}y \leq \frac{C}{\Phi(R)}\left(\frac{R}{\theta r}\right)^{n+2}. \tag{3.4.5}$$

Combining (3.4.4) and (3.4.5), we arrive at

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)}w(x) \leq \frac{C}{\Phi(R)}\left(\frac{R}{\theta r}\right)^{n+2} \quad \text{in } B_{\theta r/2}(x_0).$$

Therefore, by applying Theorem 3.3.3 to w in $B_{\theta r/2}(x_0)$, we have

$$\begin{aligned}
&|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \\
&= \left|\left\{w > \left((1-\theta/2)^{-\gamma} - 1/2\right)u(x_0)\right\} \cap B_{\theta r/4}(x_0)\right| \\
&\leq C\frac{(\theta r/4)^n}{((1-\theta/2)^{-\gamma}-1/2)^{\varepsilon}u(x_0)^{\varepsilon}}\left(w(x_0) + C\frac{\Phi(\theta r/4)}{\Phi(R)}\left(\frac{R}{\theta r}\right)^{n+2}\right)^{\varepsilon}.
\end{aligned}$$

We make the quantity $(1-\theta/2)^{-\gamma} - 1/2$ bounded away from 0 by taking $\theta > 0$ sufficiently small. Therefore, recalling that $u(x_0) = t(R/2r)^{\gamma}$ and $w(x_0) = ((1-\theta/2)^{-\gamma}-1)u(x_0)$, we obtain

$$\begin{aligned}
&|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \\
&\leq C(\theta r/4)^n\left(\left((1-\theta/2)^{-\gamma}-1\right) + t^{-1}\left(\frac{2r}{R}\right)^{\gamma}\frac{\Phi(\theta r/4)}{\Phi(R)}\left(\frac{R}{\theta r}\right)^{n+2}\right)^{\varepsilon}.
\end{aligned}$$

By using Lemma 2.1.2 and $\sigma_0 \leq \sigma_1 < 2$, we have

$$\frac{\Phi(\theta r/4)}{\Phi(R)} = \frac{\underline{C}_\varphi(R)}{\underline{C}_\varphi(\theta r/4)} \left(\frac{\theta r}{4R}\right)^2 \leq \left(\frac{\theta r}{4R}\right)^2 + a^2 \left(\frac{\theta r}{4R}\right)^{\sigma_1} \leq C \left(\frac{\theta r}{R}\right)^{\sigma_0}.$$

Since $R > r$ and $\gamma > n+2$, we obtain that

$$\left(\frac{2r}{R}\right)^\gamma \frac{\Phi(\theta r/4)}{\Phi(R)} \left(\frac{R}{\theta r}\right)^{n+2} \leq C\theta^{-n+\sigma_0-2} \left(\frac{r}{R}\right)^{\gamma-(n+2)+\sigma_0} \leq C\theta^{-n+\sigma_0-2}.$$

Therefore,

$$|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \leq C(\theta r/4)^n \left(((1-\theta/2)^{-\gamma} - 1)^\varepsilon + t^{-\varepsilon}\theta^{-n+\sigma_0-2}\right).$$

We choose a constant $\theta > 0$ sufficiently small so that

$$C(\theta r/4)^n \left((1-\theta/2)^{-\gamma} - 1\right)^\varepsilon \leq \frac{1}{4} \left|B_{\theta r/4}(x_0)\right|.$$

If $t > 0$ is sufficiently large so that $C(\theta r/4)^n t^{-\varepsilon}\theta^{-n+\sigma_0-2} \leq |B_{\theta r/4}(x_0)|/4$, then we arrive at (3.4.3). Therefore, t is uniformly bounded and the result follows. □

We remark that the Harnack inequality consists of the weak Harnack inequality and the local boundedness. The weak Harnack inequality requires u to be a nonnegative bounded supersolution, whereas the local boundedness requires u to be a subsolution. We have proved the full Harnack inequality directly, but the local boundedness is almost contained in the proof of Theorem 3.0.2. We state the local boundedness for the later use (see Lemma 4.2.3) and provide its proof for the completeness. In the following theorem, ω denotes the weight

$$\omega = \frac{1}{\underline{C}_\varphi(1)} \frac{1}{1+|y|^n\varphi(|y|)}$$

and $\|\cdot\|_{L^1(\mathbb{R}^n,\omega)}$ is the weighted L^1-norm.

Theorem 3.4.3 (Local boundedness)**.** *Assume $\sigma_1 \geq \sigma_0 > 0$. Let $u \in C(\overline{B_1})$ be a function such that $\|u\|_{L^1(\mathbb{R}^n,\omega)} \leq C_0$. If $\Phi(1)\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u \geq -C_0$ in B_1 in the* *viscosity sense, then* $u \le CC_0$ *in* $B_{1/2}$ *for some universal constant* $C > 0$, *depending only on* n, λ, Λ, a, *and* σ_0.

Proof. We may assume that $C_0 = 1$ by considering u/C_0 instead of u. Let $\varepsilon > 0$ be the constant as in Theorem 3.3.3 and let $\gamma = (n+2)/\varepsilon$. Let us consider the minimal value of $t > 0$ such that

$$u(x) \le h_t(x) := t\,(1-|x|)^{-\gamma} \quad \text{for all } x \in B_{3/4}.$$

Then there exists $x_0 \in B_{3/4}$ satisfying $u(x_0) = h_t(x_0)$. It is enough to show that t is uniformly bounded.

Let $d = 1 - |x_0|, r = d/2$, and let $A = \{u > u(x_0)/2\}$. The assumption $\|u\|_{L^1(\mathbb{R}^n,\omega)} \le 1$ implies that $\|u\|_{L^1(B_1)} \le C(n, \sigma_0)$, and hence

$$|A \cap B_1| \le C \left| \frac{2}{u(x_0)} \right| \le Ct^{-1} d^{\gamma} \le Ct^{-1} d^n.$$

Since $B_r(x_0) \subset B_1$ and $r = d/2$, we obtain

$$|\{u > u(x_0)/2\} \cap B_r(x_0)| \le Ct^{-1}|B_r(x_0)|.$$

We will show that there is a universal constant $\theta > 0$ such that

$$|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \le \frac{1}{2}|B_{\theta r/4}|,$$

provided that t is sufficiently large, which will give us a contradiction as in the proof of Theorem 3.0.2.

Let us first estimate $|\{u < u(x_0)/2\} \cap B_{\theta r}(x_0)|$ for small $\theta > 0$, which will be chosen uniformly later. For every $x \in B_{\theta r}(x_0)$, we have

$$u(x) \le h_t(x) \le t|d - \theta r|^{-\gamma} = (1 - \theta/2)^{-\gamma}\, u(x_0).$$

Let us consider the function $v(x) := (1-\theta/2)^{-\gamma}\, u(x_0) - u(x)$ and its positive part $w = v_+$. For $x \in B_{\theta r/2}(x_0)$, we have

$$\begin{aligned}\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w(x) &\leq \mathcal{M}^-_{\mathcal{L}_0(\varphi)} v(x) + \mathcal{M}^+_{\mathcal{L}_0(\varphi)} v_-(x) \\ &\leq \frac{1}{\Phi(1)} + 2\Lambda C_\varphi \int_{\{v(x+y)<0\}} \frac{v_-(x+y)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\ &\leq \frac{1}{\Phi(1)} + 2\Lambda C_\varphi \int_{\mathbb{R}^n \setminus B_{\theta r/2}} \frac{(u(x+y) - (1-\theta/2)^{-n} u(x_0))_+}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\ &\leq \frac{1}{\Phi(1)} + 2\Lambda C_\varphi \int_{\mathbb{R}^n \setminus B_{\theta r/2}(x)} \frac{|u(z)|}{|z-x|^n \varphi(|z-x|)} \,\mathrm{d}z.\end{aligned}$$

By Lemma 2.1.3, we obtain

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w(x) \leq \frac{1}{\Phi(1)} + \frac{2\Lambda c_2}{\Phi(1) \underline{C}_\varphi(1)} \int_{\mathbb{R}^n \setminus B_{\theta r/2}(x)} \frac{|u(z)|}{|z-x|^n \varphi(|z-x|)} \,\mathrm{d}z.$$

We notice that there is a constant $c > 0$ such that $|z - x| \geq c\theta r(1 + |z|)$ for all $z \in \mathbb{R}^n \setminus B_{\theta r/2}(x)$. Thus, by means of the weak scaling condition (2.1.1), we have

$$\frac{1}{|z-x|^n \varphi(|z-x|)} \leq \frac{C}{1 + |z|^n \varphi(|z|)} \frac{1}{(\theta r)^{n+2}},$$

and hence,

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w(x) \leq \frac{C}{\Phi(1)} \frac{1}{(\theta r)^{n+2}}.$$

Therefore, by Theorem 3.3.3 we obtain

$$\begin{aligned}&|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \\ &= |\{w > ((1-\theta/2)^{-n} - 1/2) u(x_0)\} \cap B_{\theta r/4}(x_0)| \\ &\leq \frac{C(\theta r/4)^n}{((1-\theta/2)^{-n} - 1/2)^\varepsilon u(x_0)^\varepsilon} \left(w(x_0) + C \frac{\Phi(\theta r/4)}{\Phi(1)} \frac{1}{(\theta r)^{n+2}} \right)^\varepsilon.\end{aligned}$$

The same argument as in the proof of Theorem 3.4.2 shows that

$$|\{u < u(x_0)/2\} \cap B_{\theta r/4}(x_0)| \leq C(\theta r/4)^n \left(\left((1-\theta/2)^{-\gamma} - 1 \right)^\varepsilon + t^{-\varepsilon} \theta^{-n+\sigma_0-2} \right).$$

By taking $\theta > 0$ sufficiently small and then taking $t > 0$ sufficiently large, we arrive at the desired contradiction. Therefore, t is uniformly bounded and the result follows. □

3.5 Hölder Estimates

In this section we prove Theorem 3.0.3 and Theorem 3.0.4. While the Hölder estimate is a trivial consequence of the Harnack inequality in the case of local operators, the Hölder estimate does not follow immediately from the Harnack inequality for a nonlocal operator. This is because the Harnack inequality requires u to be nonnegative in the whole space $\mathbb{R}^n$, not in a ball. Thus, in order to obtain the Hölder estimates, we need to investigate u outside the balls. Let us prove the following lemma, from which Theorem 3.0.3 follows by a simple covering argument.

Lemma 3.5.1. *Assume that $\sigma_1 \geq \sigma_0 > 0$. There is a universal constant $\varepsilon_1 > 0$ such that if $|u| \leq \frac{1}{2}$ in $\mathbb{R}^n$ and*

$$\Phi(R)\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u \geq -\varepsilon_1 \quad \text{and} \quad \Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)}u \leq \varepsilon_1 \quad \text{in } B_R$$

in the viscosity sense, then $u \in C^\alpha$ at the origin with an estimate

$$|u(x) - u(0)| \leq CR^{-\alpha}|x|^\alpha, \tag{3.5.1}$$

where $\alpha > 0$ and $C > 0$ are universal constants depending only on n, λ, Λ, a, and σ_0.

Proof. We will show that there exist an increasing sequence $\{m_k\}_{k\geq 0}$ and a decreasing sequence $\{M_k\}_{k\geq 0}$ satisfying $m_k \leq u \leq M_k$ in $B_{4^{-k}R}$ and $M_k - m_k = 4^{-\alpha k}$, so that (3.5.1) follows.

For $k = 0$, we choose $m_0 = -1/2$ and $M_0 = 1/2$. We now assume that we have sequences up to m_k and M_k. We want to show that we can continue the sequences by finding m_{k+1} and M_{k+1}.

In the ball $B_{4^{-(k+1)}R}$, either $u > (M_k + m_k)/2$ or $u \leq (M_k + m_k)/2$ in at least half of the points in measure. Let us say that

$$\left|\left\{u > \frac{M_k + m_k}{2}\right\} \cap B_{4^{-(k+1)}R}\right| \geq \frac{|B_{4^{-(k+1)}R}|}{2}.$$

Define the function

$$v(x) := \frac{u(x) - m_k}{(M_k - m_k)/2}.$$

Then $v \geq 0$ in $B_{4^{-k}R}$ by the induction hypothesis, and $|\{v > 1\} \cap B_{4^{-(k+1)}R}| \geq |B_{4^{-(k+1)}R}|/2$. In order to apply Theorem 3.3.3, we define $w = v^+$. Note that we have $|\{w > 1\} \cap B_{4^{-(k+1)}R}| \geq |B_{4^{-(k+1)}R}|/2$. Since $\Phi(R)\mathcal{M}^-_{\mathcal{L}_0(\varphi)} u \leq \varepsilon_1$ in B_R, we obtain

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w \leq \mathcal{M}^-_{\mathcal{L}_0(\varphi)} v + \mathcal{M}^+_{\mathcal{L}_0(\varphi)} v^- \leq \frac{2\varepsilon_1}{M_k - m_k} \frac{1}{\Phi(R)} + \mathcal{M}^+_{\mathcal{L}_0(\varphi)} v^- \quad \text{in } B_R.$$

For $\mathcal{M}^+_{\mathcal{L}_0} v^-$, we claim that

$$v(x) \geq -2\left(\frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1\right) \quad \text{in } \mathbb{R}^n \setminus B_{4^{-k}R}.$$

Indeed, for $x \in B_{4^{-k+j}R} \setminus B_{4^{-k+j-1}R}, 1 \leq j \leq k$,

$$\begin{aligned} v(x) = \frac{u(x) - m_k}{(M_k - m_k)/2} &\geq \frac{m_{k-j} - M_{k-j} + M_k - m_k}{(M_k - m_k)/2} \\ &= -2(4^{\alpha j} - 1) \geq -2\left(\frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1\right), \end{aligned}$$

and for $x \in \mathbb{R}^n \setminus B_R$,

$$\begin{aligned} v(x) \geq \frac{-\frac{1}{2} - M_k + M_k - m_k}{(M_k - m_k)/2} &= -(1 + 2M_k)4^{\alpha k} + 2 \\ &\geq -2(4^{\alpha k} - 1) \geq -2\left(\frac{|4x|^\alpha}{(4^{-k}R)^\alpha} - 1\right). \end{aligned}$$

Therefore, we have for $x \in B_{3\cdot 4^{-(k+1)}R}$

$$\begin{aligned}\mathcal{M}^+_{\mathcal{L}_0(\varphi)} v^-(x) &\le 2\Lambda C_\varphi \int_{\{v(x+y)<0\}} \frac{v^-(x+y)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\ &\le 4\Lambda C_\varphi \int_{\mathbb{R}^n \setminus B_{4^{-(k+1)}R}} \left(\frac{|4(x+y)|^\alpha}{(4^{-k}R)^\alpha} - 1 \right) \frac{\mathrm{d}y}{|y|^n \varphi(|y|)}.\end{aligned}$$

Since $|x+y| \le 3 \cdot 4^{-(k+1)}R + |y| \le 4|y|$, we obtain

$$\begin{aligned}\mathcal{M}^+_{\mathcal{L}_0(\varphi)} v^-(x) &\le 4\Lambda C_\varphi \int_{\mathbb{R}^n \setminus B_{4^{-(k+1)}R}} \left(\left(\frac{16|y|}{4^{-k}R} \right)^\alpha - 1 \right) \frac{\mathrm{d}y}{|y|^n \varphi(|y|)} \\ &= C C_\varphi \int_{4^{-(k+1)}R}^\infty \left(\left(\frac{16r}{4^{-k}R} \right)^\alpha - 1 \right) \frac{\mathrm{d}r}{r\varphi(r)} \\ &\le C C_\varphi \frac{(4^{-(k+1)}R)^{\sigma_1}}{\varphi(4^{-(k+1)}R)} \int_{4^{-(k+1)}R}^\infty \left(\left(\frac{16r}{4^{-k}R} \right)^\alpha - 1 \right) r^{-1-\sigma_1} \,\mathrm{d}r,\end{aligned}$$

where we have used the weak scaling condition (2.1.1) for the last inequality. If we have taken $\alpha < \sigma_0$, then

$$\int_{4^{-(k+1)}R}^\infty \left(\left(\frac{16r}{4^{-k}R} \right)^\alpha - 1 \right) r^{-1-\sigma_1} \,\mathrm{d}r = \left(\frac{4^\alpha}{\sigma_1 - \alpha} - \frac{1}{\sigma_1} \right) (4^{-(k+1)}R)^{-\sigma_1}.$$

Since

$$\frac{4^\alpha}{\sigma_1 - \alpha} - \frac{1}{\sigma_1} = \frac{\sigma_1(4^\alpha - 1) + \alpha}{\sigma_1(\sigma_1 - \alpha)} \le \frac{2(4^\alpha - 1) + \alpha}{\sigma_0(\sigma_0 - \alpha)} =: f(\alpha, \sigma_0),$$

we have

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)} v^-(x) \le C \frac{C_\varphi}{\varphi(4^{-(k+1)}R)} f(\alpha, \sigma_0) \quad \text{in } B_{3\cdot 4^{-(k+1)}R}.$$

Therefore, we have obtained

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)} w \le \frac{2\varepsilon_1 4^{\alpha k}}{\Phi(R)} + C \frac{C_\varphi}{\varphi(4^{-(k+1)}R)} f \quad \text{in } B_{3\cdot 4^{-(k+1)}R}.$$

Note that we have the same equation in $B_{4^{-(k+1)}R}(x)$ for $x \in B_{2\cdot 4^{-(k+1)}R}$. We apply Theorem 3.3.3 to w in $B_{4^{-(k+1)}R}(x)$, to obtain

$$\begin{aligned}\frac{|B_{4^{-(k+1)}R}|}{2} &\leq |\{w \geq 1\} \cap B_{4^{-(k+1)}R}| \\ &\leq C\left(\frac{R}{4^{k+1}}\right)^n \left(w(x) + \left(\frac{2\varepsilon_1 4^{\alpha k}}{\Phi(R)} + \frac{CC_\varphi f}{\varphi(R/4^{k+1})}\right)\Phi\left(4^{-k}R/8\right)\right)^\varepsilon.\end{aligned}$$

It follows from Lemma 2.1.2 that

$$\frac{\Phi(4^{-k}R/8)}{\Phi(R)} = \left(4^{-k}/8\right)^2 \frac{\underline{C}_\varphi(R)}{\underline{C}_\varphi(4^{-k}R/8)} \leq C4^{-2k} + C4^{-\sigma_1 k}.$$

Moreover, we have by using Lemma 2.1.3 and (2.1.3),

$$\frac{C_\varphi}{\varphi(R/4^{k+1})}\Phi(4^{-k}R/8) \leq C.$$

Therefore, using $\alpha < \sigma_0 \leq \sigma_1$, we have

$$\theta \leq w(x) + C\varepsilon_1\left(4^{(\alpha-2)k} + 4^{(\alpha-\sigma_1)k}\right) + Cf \leq w(x) + C\varepsilon_1 + Cf(\alpha, \sigma_0)$$

for some universal constant $\theta > 0$. Notice that we have $\lim_{\alpha\to 0^+} f(\alpha, \sigma_0) = 0$. If we have chosen α and ε_1 sufficiently small so that $C\varepsilon_1 \leq \theta/4$ and $Cf(\alpha, \sigma_0) \leq \theta/4$, then we have $w \geq \theta/2$ in $B_{2\cdot 4^{-(k+1)}R}$. Thus, if we set $M_{k+1} = M_k$ and $m_{k+1} = M_k - 4^{-\alpha(k+1)}$, then

$$M_{k+1} \geq u \geq m_k + \frac{M_k - m_k}{4}\theta = M_k - \left(1 - \frac{\theta}{4}\right)4^{-\alpha k} \geq m_{k+1}$$

in $B_{4^{-(k+1)}R}$.

On the other hand, if $|\{u \leq (M_k + m_k)/2\} \cap B_{4^{-(k+1)}R}| \geq |B_{4^{-(k+1)}R}|/2$, we define

$$v(x) = \frac{M_k - u(x)}{(M_k - m_k)/2},$$

and continue in the same way using that $\Phi(R)\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u \geq -\varepsilon_1$. □

As a corollary of Theorem 3.0.3, we obtain Theorem 3.0.4. Indeed, the standard arguments using the incremental quotients of solutions for $C^{1,\alpha}$ regularity estimates are provided in [15, Section 13]. We omit the proof because it is essentially the same with the proof of [15, Theorem 13.1], except that we need to use Theorem 3.0.3 instead of [15, Theorem 12.1].

Chapter 4

Evans–Krylov and Schauder-Type Estimates

This chapter is concerned with the Evans–Krylov-type and the Schauder-type generalized Hölder estimates for nonlocal fully nonlinear equations with rough kernels of variable orders. The results in this chapter are based on the joint work in [49]. We first provide the Evans–Krylov-type interior estimates for concave translation invariant elliptic equations with respect to the class $\mathcal{L}_0(\varphi)$. We next establish the Schauder-type estimates for equations having x dependence in a generalized Hölder fashion. All the regularity estimates are obtained in much finer scale of Hölder space $C^{\varphi\psi}$, and recover the classical Evans–Krylov theorem and Schauder theorem for second order fully nonlinear equations as limits. Moreover, we do not restrict ourselves to the Bellman-type operators, and consider nonlinear operators in full generality.

The Evans–Krylov-type $C^{\sigma+\alpha}$ interior estimate for nonlocal equations was first established by Caffarelli and Silvestre [17]. This result states that if u is a bounded solution of $\inf_{L\in\mathcal{L}_2} Lu = 0$ in B_1, then $\|u\|_{C^{\sigma+\alpha}(\overline{B_{1/2}})} \leq C\|u\|_{L^\infty(\mathbb{R}^n)}$. Here $\mathcal{L}_2 = \mathcal{L}_2(\sigma)$ is the class of linear operators in $\mathcal{L}_0(\sigma)$ with an additional assumption

$$[K]_{C^2(\mathbb{R}^n\setminus B_r)} \leq \Lambda(2-\sigma)r^{-n-\sigma-2} \quad \text{for all } r > 0.$$

See [54] for the Evans–Krylov estimate for nonlocal parabolic equations. In the paper [74], Serra improved this result in [17] to the equations with rough kernels. More precisely, he proved that if $u \in C^{\sigma+\alpha}(B_1) \cap C^{\alpha}(\mathbb{R}^n)$ solves $\inf_{L\in\mathcal{L}_0} Lu = 0$ in B_1, then $\|u\|_{C^{\sigma+\alpha}(\overline{B_{1/2}})} \leq C\|u\|_{C^{\alpha}(\mathbb{R}^n)}$.

Schauder estimates have been established for linear integro-differential operators [4, 32, 63, 2] and Bellman-type integro-differential operators [74, 44] in different contexts. In [74], it was shown that the proof for the Evans–Krylov estimates also works for the equations having x dependence in C^α fashion. He proved that if $u \in C^{\sigma+\alpha}(B_1) \cap C^{\alpha}(\mathbb{R}^n)$ is a solution to a non-translation invariant equation

$$\inf_{a\in\mathcal{A}} \left(\int_{\mathbb{R}^n} \delta(u, x, y) K_a(x, y) \, \mathrm{d}y + c_a(x) \right) = 0 \quad \text{in } B_1, \tag{4.0.1}$$

where $\mathcal{A}$ is some index set, K_a are kernels satisfying (3.0.1) and

$$\int_{B_{2r}\setminus B_r} |K_a(x, y) - K_a(x', y)| \, \mathrm{d}y \leq A_0|x-x'|^\alpha \frac{2-\sigma}{r^\sigma} \quad \text{for all } x, x' \in \mathbb{R}^n, r > 0, \tag{4.0.2}$$

and c_a are functions with $\|c_a\|_{C^\alpha(\overline{B_1})} \leq C_0$, then $\|u\|_{C^{\sigma+\alpha}(\overline{B_{1/2}})} \leq C(C_0 + \|u\|_{C^\alpha(\mathbb{R}^n)})$. Moreover, it was proved that if the kernels K_a additionally satisfy

$$[K_a(x, \cdot)]_{C^\alpha(\mathbb{R}^n\setminus B_r)} \leq \Lambda(2-\sigma) r^{-n-\sigma-\alpha} \quad \text{for all } r > 0, \tag{4.0.3}$$

then the uniform estimate $\|u\|_{C^{\sigma+\alpha}(\overline{B_{1/2}})} \leq C(C_0 + \|u\|_{L^\infty(\mathbb{R}^n)})$ holds for merely bounded solutions. The subclass of $\mathcal{L}_0(\sigma)$ consisting of linear operators whose kernels satisfy (4.0.3) is denoted by $\mathcal{L}_\alpha(\sigma)$.

On the other hand, the Schauder-type $C^{\sigma+\alpha}$ estimate was also established independently by Jin and Xiong [44]. They proved that bounded solutions to Bellman-type equations, with smooth kernels in $\mathcal{L}_2$ and C^α dependence on x, have uniform estimates. The assumption $\mathcal{L}_2$ is stronger than $\mathcal{L}_\alpha$, but their proof is very different from the proof of Serra.

All the aforementioned results concerning the Evans–Krylov and Schauder estimates are dealt with nonlocal equations for a fixed order of differentiability $\sigma \in (0, 2)$, except for [2], where the Schauder theory for linear integro-differential operators of variable orders is obtained through the potential theory. The aim of this chapter is to establish the generalized Evans–Krylov and Schauder $C^{\varphi\psi}$ interior estimates for general nonlocal fully nonlinear equations with rough kernels of variable orders. Our proofs are significantly different from the proof in [2] because the equations we consider are nonlinear. Moreover, as we mentioned before, we do not restrict ourselves to the Bellman-type operators (4.0.1) and provide the regularity results in full generality without assuming an explicit form of operators.

Throughout this chapter, we always assume that the function ϕ defined by $\phi(r) = \varphi(r^{-1/2})^{-1}$ is a Bernstein function and that ϕ enjoys the weak scaling condition (1.1.4). The former assumption is for a later use of results in [2]. The latter assumption implies that, as explained in Section 1.1, the function φ satisfies the weak scaling condition (2.1.1). We may assume that $\varphi(1) = 1$ by considering $\varphi(r)/\varphi(1)$ instead of $\varphi(r)$ if necessary. Let us assume finally that

$$r\varphi'(r) \leq C\varphi(r), \tag{4.0.4}$$

which is not so restrictive because it is satisfied by all the examples in Example 1.1.1.

The operator $\mathcal{I}$ we consider in this chapter may be translation invariant or non-translation invariant. We recall that $\mathcal{L}_0(\varphi)$ is the class of linear operators of the form (1.2.4) with measurable kernels K satisfying (1.2.6). However, in this chapter let $\mathcal{L}_0(\varphi)$ denote a class of linear operators of the same form (1.2.4) but with measurable kernels K satisfying

$$\lambda \frac{c_\varphi}{|y|^n \varphi(|y|)} \leq K(y) \leq \Lambda \frac{c_\varphi}{|y|^n \varphi(|y|)},$$

where

$$c_\varphi = \frac{1}{\underline{C}_\varphi(1)}.$$

We have seen in the previous chapter that the constant $C(n, \varphi)$ plays a fundamental role in obtaining robust estimates. Since the limit behavior of the estimates as $\sigma_1 \to 2$ is of interest only, we are going to use the constant c_φ instead of the full constant $C(n, \varphi)$ for the simplicity. It is to be noted that c_φ corresponds to $2 - \sigma$ in the case of the fractional Laplacian.

The operator under consideration in this chapter is two-fold. For the Evans–Krylov-type estimates, we will assume that $\mathcal{I}$ is a concave translation invariant elliptic operator with respect to the class $\mathcal{L}_0(\varphi)$. A typical example is the Bellman-type operator, but a novelty with respect to [74] and [44] is that the proof does not rely on an explicit form of the operator.

For the Schauder-type estimates, we will consider more general operators $\mathcal{I}(u, x)$ which are not necessarily translation invariant. The standard assumptions we need to impose on $\mathcal{I}$ are that $\mathcal{I}$ has an x dependence in Hölder fashion for the "freezing coefficients" step and that the model equation obtained by freezing coefficients has an appropriate regularity estimate. Recall that, in [74] and [44], the C^α dependence (4.0.2) in x variable is imposed to kernels of the operator. However, since we do not assume the explicit form of the operator, the Hölder dependence in x variable must be imposed directly to the operator (see (4.0.6)). Moreover, the x dependence of equation will be given in a generalized Hölder fashion.

The first main result in this chapter is the Evans–Krylov-type $C^{\varphi\psi}$ interior estimates for concave translation invariant nonlocal fully nonlinear equations with rough kernels of variable orders.

Theorem 4.0.1. *Let $0 < \lambda \leq \Lambda$, $a \geq 1$, and $\sigma_0 \in (0, 2)$. There is a universal constant $\bar{\alpha} \in (0, 1)$, depending only on n, λ, Λ, a, and σ_0, such that the following statement holds: let $\mathcal{I}$ be a concave translation invariant elliptic operator with respect to $\mathcal{L}_0(\varphi)$, and assume*

$$I_\varphi \subset [\sigma_0, 2), I_\psi \subset (0, \bar{\alpha}), I_{\varphi\psi} \cap \mathbb{N} = \emptyset, m_\varphi + \bar{\alpha} \notin \mathbb{N}, \text{ and } \lfloor m_\varphi + \bar{\alpha} \rfloor = \lfloor m_{\varphi\psi} \rfloor. \tag{4.0.5}$$

If $u \in C^{\varphi\psi}(B_1) \cap C^{\psi}(\mathbb{R}^n)$ *and* $f \in C^{\psi}(\overline{B_1})$ *satisfy* $\mathcal{I}u = f$ *in* B_1*, then*

$$\|u\|_{C^{\varphi\psi}(\overline{B_{1/2}})} \leq C \left(\|u\|_{C^{\psi}(\mathbb{R}^n)} + \|f\|_{C^{\psi}(\overline{B_1})} \right),$$

where C *is a universal constant depending only on* n*,* λ*,* Λ*,* a*,* σ_0*,* ψ*, and* $m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor$*.*

The non-integer assumptions in (4.0.5) are common and inevitable because of the well-known technical difficulty arising from the Hölder spaces.

As in the previous chapter, since the estimates in Theorem 4.0.1 and upcoming theorems are robust, the results recover the classical Evans–Krylov and Schauder theorems for second order fully nonlinear equations as limits. Notice that in this case the dependence on $m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor$ is absorbed into the dependence on ψ. Our main theorems provide new results even in the case of second order fully nonlinear equations since the data is given in the generalized Hölder sense. Moreover, the results are also new in the case of the fractional Laplacian-type equations ($\varphi(r) = r^{\sigma}$) because we do not restrict ourselves to Bellman-type operators.

The next result is the Schauder-type $C^{\varphi\psi}$ estimates for non-translation invariant fully nonlinear equations with rough kernels of variable orders. As we mentioned above, we need to impose C^{ψ} dependence in x variable directly to the operator $\mathcal{I}$. For this purpose, we consider, for a fixed point z, the function

$$\beta_{\mathcal{I}}(x, x') = \sup \frac{|\mathcal{I}(u, x) - \mathcal{I}(u, x')|}{\|u\|'_{C^{\varphi\psi}(\overline{B_r(z)})} + \|u\|_{L^\infty(\mathbb{R}^n)}}, \quad x, x' \in \overline{B_r(z)},$$

where the supremum is taken over the space of all nontrivial functions with $\|u\|'_{C^{\varphi\psi}(\overline{B_r(z)})} + \|u\|_{L^\infty(\mathbb{R}^n)} < +\infty$. The function $\beta_{\mathcal{I}}$ is a nonlocal analogue of the function β_F in [18], and it measures the oscillation of $\mathcal{I}$ in the x variable. We define $\mathcal{I}_z$ by $\mathcal{I}_z u(x) := \mathcal{I}(\tau_{x-z}u, z)$, where $\tau_z u(x) = u(x+z)$, which is an operator obtained by freezing coefficients of the operator $\mathcal{I}$. Note that $\mathcal{I}_z$ is translation invariant since $\mathcal{I}_z \tau_w u(x) = \mathcal{I}(\tau_{x-z}\tau_w u, z) = \mathcal{I}(\tau_{x+w-z}u, z) =$

$\mathcal{I}_z u(x+w)$. The closedness of the operators $\mathcal{I}$ and $\mathcal{I}_z$ in the C^ψ fashion is given by

$$\beta_{\mathcal{I}-\mathcal{I}_z}(x,x') \le A_0 \psi(|x-x'|) \quad \forall x,x' \in B_r(z), \quad \text{for every ball } B_r(z) \subset B_1. \tag{4.0.6}$$

Notice that (4.0.6) corresponds to $[a_{ij}(\cdot) - a_{ij}(z)]_{C^\psi(\overline{B_r(z)})} \le A_0$, or equivalently, $[a_{ij}]_{C^\psi(\overline{B_r(z)})} \le A_0$, in the case of second order linear operator in a non-divergence form.

Another assumption we need is the regularity estimates for the model equations. We say that $\mathcal{I}_z$ satisfies the *Evans–Krylov-type estimates in* $B_r = B_r(z)$ if, for given $\alpha \in (0, m_\psi)$ and given functions $f \in C^\psi(\overline{B_r})$ and $v \in C^{\varphi\psi}(B_r) \cap C^\psi(\mathbb{R}^n)$, $u \in C^{\varphi+\alpha}(\mathbb{R}^n)$ solves the equation $\mathcal{I}_z(u+v) = f$ in B_r, then $u \in C^{\varphi\psi}(\overline{B_{r/2}})$ and

$$[u]_{C^{\varphi\psi}(\overline{B_{r/2}})} \le C \left(\|u\|_{C^{\varphi+\alpha}(\mathbb{R}^n)} + [f]_{C^\psi(\overline{B_r})} + \sup_{L \in \mathcal{L}} [Lv]_{C^\psi(\overline{B_r})} \right) \tag{4.0.7}$$

for some universal constant C. The class $\mathcal{L}$ in (4.0.7) is $\mathcal{L}_0(\varphi)$ or $\mathcal{L}_\psi(\varphi)$ according to the operator $\mathcal{I}$, i.e., if $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_0(\varphi)$ or $\mathcal{L}_\psi(\varphi)$, then $\mathcal{L} = \mathcal{L}_0(\varphi)$ or $\mathcal{L} = \mathcal{L}_\psi(\varphi)$, respectively.

We point out that we say that $\mathcal{I}_z$ satisfies the Evans–Krylov-type estimate if solutions enjoy (4.0.7), not the estimate in Theorem 4.0.1. This is because the estimate of the form (4.0.7) is useful for later uses in the following theorems, as well as it implies the estimate in Theorem 4.0.1 (see Proposition 4.3.1). The concave non-translation invariant elliptic operators are, of course, examples of operators satisfying the Evans–Krylov-type estimates.

Theorem 4.0.2. *Let $\bar\alpha$ be the constant in Theorem 4.0.1, and assume* (4.0.5). *Let $\mathcal{I}$ be a non-translation invariant operator which is elliptic with respect to $\mathcal{L}_0(\varphi)$ and satisfies* (4.0.6). *Suppose that $\mathcal{I}_z$ satisfies the Evans–Krylov-type estimates in $B_r(z)$ for every ball $B_r(z) \subset B_1$. If $u \in C^{\varphi\psi}(B_1) \cap C^\psi(\mathbb{R}^n)$ and*

$f \in C^{\psi}(\overline{B_1})$ satisfy $\mathcal{I}(u,x) = f(x)$ in B_1, then

$$\|u\|_{C^{\varphi\psi}(\overline{B_{1/2}})} \leq C \left(\|u\|_{C^{\psi}(\mathbb{R}^n)} + \|f\|_{C^{\psi}(\overline{B_1})} \right),$$

where C is a universal constant depending only on n, λ, Λ, a, σ_0, A_0, ψ, and $m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor$.

We will see, in Section 4.5, that the uniform estimates in Theorem 4.0.1 and Theorem 4.0.2 would be false if solutions u are assumed to be merely bounded, as in [74]. However, if the operator $\mathcal{I}$ is elliptic with respect to the subclass $\mathcal{L}_{\psi}(\varphi) \subset \mathcal{L}_0(\varphi)$ of linear operators whose kernels satisfy

$$[K]_{C^{\psi}(\mathbb{R}^n \setminus B_r)} \leq \Lambda \frac{c_{\varphi}}{r^n \varphi(r) \psi(r)} \quad \text{for all } r > 0, \tag{4.0.8}$$

then we obtain $C^{\varphi\psi}$ uniform estimates for merely bounded solutions. Notice that the condition (4.0.8) generalizes (4.0.3).

Theorem 4.0.3. *Suppose that $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_{\psi}(\varphi)$ and satisfies the same assumptions as in Theorem 4.0.2. If $u \in C^{\varphi\psi}(B_1) \cap L^{\infty}(\mathbb{R}^n)$ and $f \in C^{\psi}(\overline{B_1})$ satisfy $\mathcal{I}(u,x) = f(x)$ in B_1, then*

$$\|u\|_{C^{\varphi\psi}(\overline{B_{1/2}})} \leq C \left(\|u\|_{L^{\infty}(\mathbb{R}^n)} + \|f\|_{C^{\psi}(\overline{B_1})} \right),$$

where C is a universal constant depending only on n, λ, Λ, a, σ_0, A_0, ψ, and $m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor$.

The idea of proofs is based on a Liouville-type theorem and a compactness argument using blowup sequences. This argument has been used successfully to establish regularity theory for nonlocal equations. See, for examples, [75], [64], and [65]. The argument heavily relies on the scale invariance of the equations because it allows us to consider blowup sequences and its limit. However, the kernels of operators in $\mathcal{L}_0(\varphi)$ are not homogeneous and hence our equations do not have the scale invariance. Main difficulty arises at this point in the scaling argument. We will see that the rescaled equation and rescaled solution are related to a new scale function, that is, rescaled ones behave differently at each scale. Even though the rescaled equation may be different from the original one, the weak scaling condition will make the rescaled equations belong to the same class of equations with different scale functions, but with the same constants σ_1, σ_2, and a, and the same ellipticity constants λ and Λ. In other words, the weak scaling condition makes the rescaling procedure preserve the key features of the equations and solutions.

The chapter is organized as follows. In Section 4.1, we observe how the weak scaling condition serves to rescale the equations and solutions. In Section 4.2, the Liouville-type theorem is stated and proved, which will be the key ingredient of the proof of the Evans–Krylov-type theorem. Section 4.3 is devoted to the proof of Theorem 4.0.1, where the compactness arguments with blowup sequences are presented. By using the Evans–Krylov-type estimates and by freezing coefficients of the equations, we establish the Schauder-type estimates in Section 4.4. Both Theorem 4.0.2 and Theorem 4.0.3 will be proved in this section. We finish this chapter with counterexamples to $C^{\varphi\psi}$ interior regularity for merely bounded solutions in Section 4.5.

4.1 Scaling with Varying Scale

In this section we study how the rescaling procedure works. Let φ satisfy the weak scaling condition (2.1.1) with constant $0 < \sigma_1 \le \sigma_2 < 2$ and $a \ge 1$. We first observe in the following proposition that if some equation is related to φ, then a rescaled equation is related to a new scale function $\bar{\varphi}$ which is defined, for given $\rho > 0$, by

$$\bar{\varphi}(r) = \frac{\varphi(\rho r)}{\varphi(\rho)}. \tag{4.1.1}$$

This means that the rescaling argument may break since the rescaled equation is not the same with the original equation. However, since $\bar{\varphi}$ satisfies the weak scaling condition with the same constants σ_1, σ_2, and a, the rescaled equation and the original equation are of the same type. Thus, the following proposition can be used in obtaining uniform estimates that depend on the constants σ_1, σ_2, and a, but not on φ itself.

Proposition 4.1.1. *If $\mathcal{I}$ is elliptic with respect to $\mathcal{L}_0(\varphi)$, then the operator $\bar{\mathcal{I}}$, defined by*

$$\bar{\mathcal{I}}(\bar{u},\bar{x}) := \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\mathcal{I}\left(\bar{u}((\cdot - z)/\rho), z+\rho\bar{x}\right),$$

is elliptic with respect to $\mathcal{L}_0(\bar{\varphi})$ with the same ellipticity constants. Moreover, if u is a solution of $\mathcal{I}(u,x) = f(x)$ in $B_\rho(z)$, then the function $\bar{u}$, defined by $\bar{u}(\bar{x}) = u(z+\rho\bar{x})$, solves $\bar{\mathcal{I}}(\bar{u},\bar{x}) = \bar{f}(\bar{x})$ in B_1, where

$$\bar{f}(\bar{x}) = \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}f(z+\rho\bar{x}).$$

Proof. For the first assertion, it is enough to show that

$$\begin{aligned}\mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi})}\bar{u}(\bar{x}) &= \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u(x) \quad \text{and}\\ \mathcal{M}^-_{\mathcal{L}_0(\bar{\varphi})}\bar{u}(\bar{x}) &= \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\mathcal{M}^-_{\mathcal{L}_0(\varphi)}u(x),\end{aligned} \tag{4.1.2}$$

where $x = z+\rho\bar{x}$. This follows from the simple change of variables: for $y = \rho\bar{y}$,

$$\begin{aligned}\mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi})}\bar{u}(\bar{x}) &= \int_{\mathbb{R}^n}\left(\Lambda\delta^+(\bar{u},\bar{x},\bar{y}) - \lambda\delta^-(\bar{u},\bar{x},\bar{y})\right)\frac{c_{\bar{\varphi}}}{|\bar{y}|^n\bar{\varphi}(|\bar{y}|)}\,\mathrm{d}\bar{y}\\ &= \int_{\mathbb{R}^n}\left(\Lambda\delta^+(\bar{u},\bar{x},y/\rho) - \lambda\delta^-(\bar{u},\bar{x},y/\rho)\right)\frac{c_{\bar{\varphi}}}{|y|^n\varphi(|y|)/\varphi(\rho)}\,\mathrm{d}y\\ &= \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\int_{\mathbb{R}^n}\left(\Lambda\delta^+(u,x,y) - \lambda\delta^-(u,x,y)\right)\frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y\\ &= \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u(x),\end{aligned}$$

and the same argument holds for $\mathcal{M}^-$. Thus $\bar{\mathcal{I}}$ is elliptic with respect to $\mathcal{L}_0(\bar{\varphi})$. The second assertion is obvious. $\square$

When we rescale the equation for a while to apply known estimates and then rescale back, Proposition 4.1.1 is very useful. However, it is not sufficient for the blowup sequence argument. For this purpose, we study the limit behavior of the scale function (4.1.1) as $\rho \to 0$.

Lemma 4.1.2. *Let φ satisfy the weak scaling condition and* (4.0.4)*. Let $\{\rho_j\}$ be a sequence such that $\rho_j \searrow 0$ as $j \to \infty$, and set $\bar{\varphi}_j(r) = \varphi(\rho_j r)/\varphi(\rho_j)$. Then $\bar{\varphi}_j$ converges locally uniformly to some function $\bar{\varphi}$ that satisfies the weak scaling condition with the same constants.*

Proof. Fix $\varepsilon > 0$ and let $\delta > 0$ be a small constant to be determined. For $|x - x_0| < \delta$, we have

$$|\bar{\varphi}_j(x) - \bar{\varphi}_j(x_0)| = \frac{|\varphi(\rho_j x) - \varphi(\rho_j x_0)|}{\varphi(\rho_j)} = \frac{\varphi'(\rho_j x_0^*)}{\varphi(\rho_j)} |\rho_j x - \rho_j x_0|$$

for some point x_0^* lying in between x_0 and x. By the assumption (4.0.4) and the weak scaling condition, we obtain

$$\frac{\varphi'(\rho_j x_0^*)}{\varphi(\rho_j)} |\rho_j x - \rho_j x_0| \leq \frac{\varphi(\rho_j x_0^*)}{\varphi(\rho_j)} \frac{|x - x_0|}{x_0^*} \leq a \max\{(x_0^*)^{\sigma_1 - 1}, (x_0^*)^{\sigma_2 - 1}\}\delta.$$

By taking δ sufficiently small so that $a \max\{(x_0^*)^{\sigma_1 - 1}, (x_0^*)^{\sigma_2 - 1}\}\delta < \varepsilon$, we conclude that $\{\bar{\varphi}_j\}$ is equicontinuous. Moreover, the weak scaling condition shows that $\{\bar{\varphi}_j\}$ is locally uniformly bounded. Therefore, by the Arzelà–Ascoli theorem and the diagonal sequence argument, we find a subsequence of $\{\bar{\varphi}_j\}$, still denoted by $\{\bar{\varphi}_j\}$, that converges locally uniformly to some function $\bar{\varphi}$. The function $\bar{\varphi}$ enjoys the weak scaling condition with the same constants. Indeed, we see that for $0 < r \leq R$,

$$\frac{\bar{\varphi}(R)}{\bar{\varphi}(r)} = \frac{\lim_{j \to \infty} \bar{\varphi}_j(R)}{\bar{\varphi}(r)} \leq a \frac{\lim_{j \to \infty} \bar{\varphi}_j(r)}{\bar{\varphi}(r)} \left(\frac{R}{r}\right)^{\sigma_2} = a \left(\frac{R}{r}\right)^{\sigma_2}.$$

The lower bound of $\bar{\varphi}(R)/\bar{\varphi}(r)$ is obtained in the same way. □

As we observed in the previous section, the rescaled operator and rescaled function are related to a new scale function (4.1.1). The following lemmas show how the norms of rescaled functions are related to the norms of the original function.

Lemma 4.1.3. *Let $u \in C^{\psi}(\overline{B_\rho(z)})$ and define $\bar{u}(\bar{x}) = u(z+\rho\bar{x})/\psi(\rho)$. Assume that $I_\psi \cap \mathbb{N} = \emptyset$ and set $\bar{\psi}(r) = \psi(\rho r)/\psi(\rho)$ for given $\rho > 0$. Then $\bar{u} \in C^{\bar{\psi}}(\overline{B_1})$ and $[\bar{u}]_{\bar{\psi};B_1} = [u]_{\psi;B_\rho(z)}$.*

Proof. Let $k = \lfloor m_\psi \rfloor$ be the integer part of m_ψ. Since $I_\psi = I_{\bar{\psi}}$, k is also the integer part of $\bar{\psi}$. Thus, we have

$$\begin{aligned}
[\bar{u}]_{\bar{\psi};B_1} &= \sup_{\bar{x},\bar{y}\in B_1} \frac{|D^k\bar{u}(\bar{x}) - D^k\bar{u}(\bar{y})|}{\bar{\psi}(|\bar{x}-\bar{y}|)|\bar{x}-\bar{y}|^{-k}} \\
&= \frac{\rho^k}{\psi(\rho)} \sup_{\bar{x},\bar{y}\in B_1} \frac{|D^k u(z+\rho\bar{x}) - D^k u(z+\rho\bar{y})|}{\psi(|\rho\bar{x}-\rho\bar{y}|)|\rho\bar{x}-\rho\bar{y}|^{-k}} \rho^{-k}\psi(\rho) \\
&= \sup_{x,y\in B_\rho(z)} \frac{|D^k u(x) - D^k u(y)|}{\psi(|x-y|)|x-y|^{-k}} = [u]_{\psi;B_\rho(z)} < +\infty,
\end{aligned}$$

where $x = z+\rho\bar{x}$ and $y = z+\rho\bar{y}$. Since $\|\bar{u}\|_{C^0(\overline{B_1})} = \frac{1}{\psi(\rho)}\|u\|_{C^0(\overline{B_\rho(z)})} < +\infty$, by Lemma 2.2.2, we obtain $\bar{u} \in C^{\bar{\psi}}(\overline{B_1})$ with $[\bar{u}]_{\bar{\psi};B_1} = [u]_{\psi;B_\rho(z)}$. □

Lemma 4.1.4. *Let $u \in C^{\psi}(\overline{B_\rho(z)})$ and define $\bar{u}(\bar{x}) = u(z+\rho\bar{x})$. Assume that $I_\psi \cap \mathbb{N} = \emptyset$ and set $\bar{\psi}(r) = \psi(\rho r)/\psi(\rho)$ for given $\rho > 0$. Then $\bar{u} \in C^{\bar{\psi}}(\overline{B_1})$ and $\|\bar{u}\|'_{\bar{\psi};B_1} = \|u\|'_{\psi;B_\rho(z)}$.*

Proof. As in the proof of Lemma 4.1.3, let $k = \lfloor m_\psi \rfloor$. Then, we have

$$\begin{aligned}
\|\bar{u}\|'_{\bar{\psi};B_1} &= \sum_{i=0}^{k} 2^i \|D^i\bar{u}\|_{C^0(\overline{B_1})} + \bar{\psi}(2) \sup_{\bar{x},\bar{y}\in B_1} \frac{|D^k\bar{u}(\bar{x}) - D^k\bar{u}(\bar{y})|}{\bar{\psi}(|\bar{x}-\bar{y}|)|\bar{x}-\bar{y}|^{-k}} \\
&= \sum_{i=0}^{k} (2\rho)^i \|D^i u\|_{C^0(\overline{B_\rho(z)})} + \frac{\psi(2\rho)}{\psi(\rho)} \sup_{x,y\in B_\rho(z)} \frac{|D^k u(x) - D^k u(y)|}{\psi(|x-y|)|x-y|^{-k}}\psi(\rho) \\
&= \|u\|'_{\psi;B_\rho(z)},
\end{aligned}$$

where $x = z+\rho\bar{x}$ and $y = z+\rho\bar{y}$, which gives the desired result. □

4.2 Liouville-Type Theorem

In this section, we prove the Liouville-type theorem that is the key ingredient of the proof of the Evans–Krylov theorem. Before we state and prove the Liouville-type theorem, we observe that if (4.0.5) is assumed for some $\bar{\alpha} \in (0,1)$, then there is $\alpha \in (0, m_\psi)$ such that

$$\lfloor m_{\varphi+\bar{\alpha}} \rfloor = \lfloor m_{\varphi\psi} \rfloor < m_{\varphi+\alpha} < m_{\varphi\psi}. \tag{4.2.1}$$

Indeed, $\alpha := m_\psi - (m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor)/2 \in (0, m_\psi)$ satisfies (4.2.1). Whenever we take $\alpha \in (0, m_\psi)$ in this chapter, we have in mind this constant.

Theorem 4.2.1. *Let $0 < \lambda \leq \Lambda$, $a \geq 1$, and $\sigma_0 \in (0,2)$. There is a universal constant $\bar{\alpha} \in (0,1)$, depending only on n, λ, Λ, a, and σ_0, such that the following statement holds: assume* (4.0.5) *and let $\alpha \in (0, m_\psi)$ be the constant satisfying* (4.2.1)*. Suppose that $u \in C^{\varphi+\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ satisfies the following properties.*

(i) There is a constant $C_1 > 0$ such that

$$\|u\|'_{\varphi+\alpha; B_R} \leq C_1 \varphi(R)\psi(R) \quad \text{for all } R \geq 1.$$

(ii) For any $h \in \mathbb{R}^n$, we have

$$\mathcal{M}^-_{\mathcal{L}_0(\varphi)}(u(\cdot+h) - u) \leq 0 \leq \mathcal{M}^+_{\mathcal{L}_0(\varphi)}(u(\cdot+h) - u) \quad \text{in } \mathbb{R}^n.$$

(iii) For every nonnegative $L^1(\mathbb{R}^n)$ function μ with compact support, we have

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)}\left(⨍ u(\cdot+h)\mu(h)\,\mathrm{d}h - u \right) \geq 0 \quad \text{in } \mathbb{R}^n,$$

where the symbol $⨍$ means the averaged integral $(\int \mu(h)\,\mathrm{d}h)^{-1} \int$.

Then u is a polynomial of degree $d := \lfloor m_{\varphi\psi} \rfloor$.

As pointed out in [74], the growth condition (i) is not enough to have $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u$ and $\mathcal{M}^-_{\mathcal{L}_0(\varphi)}u$ well defined in the classical sense, but it guarantees that $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}$ and $\mathcal{M}^-_{\mathcal{L}_0(\varphi)}$ of $u(\cdot+h)-u$ (and of $\fint u(\cdot+h)\mu(h)\,\mathrm{d}h-u$) are well defined in the classical sense.

The proof of Theorem 4.2.1 basically follows the lines of the proof of [74, Theorem 2.1], but we need to be careful in the scaling argument since the rescaled functions have different scales as we already observed in Section 4.1. The following lemma shows this scaling procedure.

Lemma 4.2.2. *Under the same setting as in Theorem 4.2.1, the rescaled function* $\bar{u}(\bar{x}) = \frac{1}{\varphi(\rho)\psi(\rho)}u(\rho\bar{x})$ *satisfies the same assumptions (i), (ii), and (iii), with* φ *and* ψ *replaced by* $\bar{\varphi}$ *and* $\bar{\psi}$*, respectively, where* $\bar{\varphi}(r) = \frac{\varphi(\rho r)}{\varphi(\rho)}$ *and* $\bar{\psi}(r) = \frac{\psi(\rho r)}{\psi(\rho)}$.

Proof. By Lemma 4.1.4, we have

$$\|\bar{u}\|'_{\bar{\varphi}+\alpha;B_R} = \frac{1}{\varphi(\rho)\psi(\rho)}\|u\|_{\varphi+\alpha;B_{\rho R}} \le C_1\frac{\varphi(\rho R)\psi(\rho R)}{\varphi(\rho)\psi(\rho)} = C_1\bar{\varphi}(R)\bar{\psi}(R),$$

which gives (i). The assumptions (ii) and (iii) follow from (4.1.2). $\square$

The key step for the proof of Theorem 4.2.1 is to prove that $L_\varphi u \in C^{\bar{\alpha}}(\overline{B_{1/2}})$, where L_φ is a linear operator of the form (1.2.4) with the kernel $K_\varphi(y) := \frac{c_\varphi}{|y|^n\varphi(|y|)}$. We will prove that $P \le C|h|^{\bar{\alpha}}$ and $N \le C|h|^{\bar{\alpha}}$ for all $h \in \overline{B_1}$, where

$$\begin{aligned} P(h) &:= \int_{\mathbb{R}^n} (\delta(u,h,y)-\delta(u,0,y))_+ \frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y \quad \text{and} \\ N(h) &:= \int_{\mathbb{R}^n} (\delta(u,h,y)-\delta(u,0,y))_- \frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y, \end{aligned}$$

and that the same proof also works when the point 0 in the above definition is replaced by any points in $\overline{B_{1/2}}$, for some constant C independent of x. Then it follows that $|L_\varphi u(x+h)-L_\varphi u(x)| \le C|h|^{\bar{\alpha}}$ for all $x \in \overline{B_{1/2}}$ and $h \in \overline{B_1}$. We point out that the constant C here is not necessarily independent of σ_1 and σ_2.

Lemma 4.2.3. *Under the same setting as in Theorem 4.2.1, there is a constant $C > 0$ such that $P(x) \le C|x|^{\bar{\alpha}}$ and $N(x) \le C|x|^{\bar{\alpha}}$ for all $x \in \overline{B_1}$.*

Proof. We first claim that, for all $R \geq 1$,

$$0 \le P \le C\psi(R) \quad \text{and} \quad 0 \le N \le C\psi(R) \quad \text{in } B_R. \tag{4.2.2}$$

Let us provide the proof of (4.2.2) for the most delicate case $d = 2$. The other cases $d = 0$ and $d = 1$ are obtained in a very similar way.

Let $x_1 = x \in B_1$ and $x_2 = 0$. If $y \in B_1$, then we have

$$|\delta(u, x_1, y) - \delta(u, x_2, y)| \le |y|^2 |D^2 u(x_1^*) - D^2 u(x_2^*)|,$$

where x_i^*, $i = 1, 2$, is a point lying in between $x_i + y$ and $x_i - y$. By the assumption (i), we obtain

$$\begin{aligned} |D^2 u(x_1^*) - D^2 u(x_2^*)| &\le [u]_{\varphi+\alpha; B_3} \varphi(|x_1^* - x_2^*|)|x_1^* - x_2^*|^{\alpha-2} \\ &\le C_1 \psi(3) 3^{-\alpha} \varphi(|x_1^* - x_2^*|)|x_1^* - x_2^*|^{\alpha-2}. \end{aligned}$$

Recall that α was chosen so that (4.2.1) holds, which yields that $\varphi(r)r^{\alpha-2}$ is almost increasing. Since $|x_1^* - x_2^*| \le 3$, we have $\varphi(|x_1^* - x_2^*|)|x_1^* - x_2^*|^{\alpha-2} \le c^{-1}\varphi(3)3^{\alpha-2} \le ac^{-1}3^{\alpha}$, with the help of the weak scaling condition (2.1.1). Thus, we have

$$|\delta(u, x, y) - \delta(u, 0, y)| \le C|y|^2. \tag{4.2.3}$$

On the other hand, if $y \in \mathbb{R}^n \setminus B_1$, then by the assumption (i) and the weak scaling condition (2.1.1), we obtain

$$\begin{aligned} |\delta(u, x, y) - \delta(u, 0, y)| &\le 4[u]_{1; B_{2|y|}} |x| \\ &\le 4C_1 \varphi(|2y|)\psi(|2y|)|2y|^{-1} \\ &\le C\varphi(|y|)\psi(|2y|)|y|^{-1}. \end{aligned} \tag{4.2.4}$$

It follows from (4.2.3), (4.2.4), and (2.1.3), that

$$\begin{aligned} P(x) &\leq C\int_{B_1} |y|^2 \frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y + C\int_{\mathbb{R}^n\setminus B_1} \varphi(|y|)\psi(|2y|)|y|^{-1}\frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\ &\leq C + C\int_1^\infty \psi(2r)r^{-2}\,\mathrm{d}r. \end{aligned}$$

Since $M_\psi < \bar{\alpha}$, by definition of M_ψ, there is a small constant $\varepsilon \geq 0$ such that $M_\psi + \varepsilon < \bar{\alpha}$ and $r \to \psi(r)r^{-M_\psi - \varepsilon}$ is almost decreasing. Thus, for $r \geq 1$, we have $\psi(r) \leq Cr^{M_\psi+\varepsilon}$, and hence $\psi(r)r^{-2}$ is integrable in $[1,\infty)$. Therefore, we arrive at $P \leq C$ in B_1, which proves the claim (4.2.2) for $R=1$. To prove (4.2.2) for all $R \geq 1$, we consider the rescaled function $\bar{u}(\bar{x}) = \frac{1}{\varphi(\rho)\psi(\rho)}u(\rho\bar{x})$ for $\rho = R$. Then Lemma 4.2.2 shows that $\bar{u}$ satisfies the assumptions (i), (ii), and (iii), with the same constant C_1, but with φ and ψ replaced by $\bar{\varphi}$ and $\bar{\psi}$, respectively. By the same argument above, we have

$$\bar{P}(\bar{x}) := \int_{\mathbb{R}^n} (\delta(\bar{u},\bar{x},\bar{y}) - \delta(\bar{u},0,\bar{y}))_+ \frac{c_{\bar{\varphi}}}{|\bar{y}|^n\bar{\varphi}(|\bar{y}|)}\,\mathrm{d}\bar{y} \leq C \quad \text{for } \bar{x} \in B_1.$$

Scaling back and using (2.1.3), we arrive at

$$\begin{aligned} P(x) &= \int_{\mathbb{R}^n} (\delta(u,x,y) - \delta(u,0,y)))_+ \frac{c_\varphi}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\ &= \frac{c_\varphi}{c_{\bar{\varphi}}}\psi(\rho)\int_{\mathbb{R}^n} (\delta(\bar{u},\bar{x},\bar{y}) - \delta(\bar{u},0,\bar{y}))_+ \frac{c_{\bar{\varphi}}}{|\bar{y}|^n\bar{\varphi}(|\bar{y}|)}\,\mathrm{d}\bar{y} \\ &\leq a^2\frac{2-\sigma_1}{2-\sigma_2}\psi(\rho)\bar{P}(\bar{x}) \leq C\psi(\rho) \quad \text{for } x \in B_\rho, \end{aligned}$$

and this proves (4.2.2) for P. The bound for N in (4.2.2) can be obtained in a similar way. Note that the constant C may depend on σ_1 and σ_2, but this constant is not relevant to the uniform estimates in the main theorems.

We have proved that

$$0 \leq P \leq C\psi(2^k) \leq C2^{k\bar{\alpha}} \quad \text{in } B_{2^k}(0) \tag{4.2.5}$$

for all $k \geq 0$, since $\psi(2^k) \leq C(2^k)^{M_\psi+\varepsilon} \leq C2^{k\bar{\alpha}}$. We next prove that (4.2.5) holds for all $k \leq 0$, from which the lemma will follow. If we show that $P \leq (1-\theta)C$ in $B_{1/2}$, then (4.2.5) for all k will follow by scaling and iteration argument. Dividing u by C, let us assume that $P \leq 1$ in B_1 and show that $P \leq 1-\theta$ in $B_{1/2}$.

Let $x_0 \in B_{1/2}$ be a point where the supremum of P in $B_{1/2}$ is attained, and define the set

$$A = \{y \in \mathbb{R}^n : \delta(u, x, y) - \delta(u, 0, y) > 0\}.$$

We define

$$v(x) := \int_A (\delta(u, x, y) - \delta(u, 0, y)) \frac{c_\varphi}{|y|^n \varphi(|y|)} \, \mathrm{d}y$$

and define the set $D := \{x \in B_1 : v \geq (1-\bar{\theta})\}$, where $\bar{\theta} = \lambda/(4\Lambda)$. We claim that there is a small constant $\eta > 0$ such that

$$|D| \leq (1-\eta)|B_1|. \tag{4.2.6}$$

Assume to the contrary that $|D| > (1-\eta)|B_1|$ for some small constant η to be determined later. Let us consider the function w given by

$$w(x) := \int_{\mathbb{R}^n \setminus A} (\delta(u, x, y) - \delta(u, 0, y)) \frac{c_\varphi}{|y|^n \varphi(|y|)} \, \mathrm{d}y.$$

We approximate $\chi_{\mathbb{R}^n \setminus A}(y) \frac{c_\varphi}{|y|^n \varphi(|y|)}$ by L^1 functions μ with compact support, use the assumption (iii), and then use the stability result [16, Lemma 4.3] to obtain that

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)} w \geq 0 \quad \text{in } \mathbb{R}^n. \tag{4.2.7}$$

Moreover, using the relation $v+w = P-N$, we have $0 \leq P-v \leq 1-(1-\bar{\theta}) = \bar{\theta}$ in D. Since the assumption (ii) shows that P and N are comparable, i.e., $\frac{\lambda}{\Lambda}P \leq N \leq \frac{\Lambda}{\lambda}P$, we obtain

$$w = (P-v) - N \leq \bar{\theta} - \frac{\lambda}{\Lambda}P \leq \bar{\theta} - \frac{\lambda}{\Lambda}(1-\bar{\theta}) \leq -\frac{\lambda}{2\Lambda} =: -c \quad \text{in } D. \tag{4.2.8}$$

Let us consider the function $\bar{w} = (w(r\cdot) + c)_+$ with $r > 0$ small. Then by (4.2.7), we have $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}\bar{w} \geq 0$ in $\mathbb{R}^n$ and we can make $\|\bar{w}\|_{L^1(\mathbb{R}^n,\omega)}$ as small as we want by taking r and η sufficiently small. Indeed, (4.2.8) gives $\bar{w} = 0$ in D/r, which covers most of $B_{1/r}$, and (4.2.2) allows us to control the weighted integral outside the ball $B_{1/r}$. Applying Theorem 3.4.3 to $\bar{w}$, we obtain $w(0) + c = \bar{w}(0) \leq c/2$, which yields a contradiction since $w(0) = 0$ by definition. Therefore, (4.2.6) holds for some small constant $\eta > 0$.

We use the assumption (iii) and [16, Lemma 4.3] again to approximate $\chi_A(y)\frac{c_\varphi}{|y|^n\varphi(|y|)}$ by L^1 functions μ with compact support. Consequently, we have $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}v \geq 0$ in $\mathbb{R}^n$. By following the computation in the proof of Lemma 3.5.1 and using (4.2.2), for given $\delta_0 > 0$ we can find $\bar{\alpha} \in (0,1)$ small enough so that $\mathcal{M}^-_{\mathcal{L}_0(\varphi)}\bar{v} \leq \delta_0$ in $B_{3/4}$, where $\bar{v} = (1-v)_+$. Therefore, by applying Theorem 3.3.3 to $\bar{v}$ and using (4.2.6), we obtain that

$$\eta|B_1| \leq |\{(1-v)_+ > \bar{\theta}\}| \leq C(1-v) \quad \text{in } B_{1/2},$$

which concludes that $v \leq 1 - \eta/C =: 1 - \theta$ in $B_{1/2}$. □

To finish the proof of Theorem 4.2.1, we make use of the potential theory [2] for linear integro-differential operators. The result [2] only provides the estimates which are not robust with respect to the order of differentiability, but it is sufficient to conclude Theorem 4.2.1.

Proof of Theorem 4.2.1. By Lemma 4.2.3, we have $L_\varphi u \in C^{\bar{\alpha}}(\overline{B_{1/2}})$ and $\|L_\varphi u\|_{\bar{\alpha};B_{1/2}} \leq C$. Thus, the potential theory shows that $\|u\|_{\varphi+\bar{\alpha};B_{1/4}} \leq C$, and in particular, $[u]_{\varphi+\bar{\alpha};B_{1/4}} \leq C$. We observed in Lemma 4.2.2 that the rescaled function $\bar{u}(\bar{x}) = \frac{1}{\varphi(\rho)\psi(\rho)}u(\rho\bar{x})$ satisfies the assumptions (i), (ii), and (iii), with φ and ψ replaced by $\bar{\varphi}$ and $\bar{\psi}$, respectively. Thus, the same argument above is applied to $\bar{u}$, resulting in $[\bar{u}]_{\bar{\varphi}+\bar{\alpha};B_{1/4}} \leq C$. Scaling back, we arrive at $[u]_{\varphi+\bar{\alpha};B_{\rho/4}} \leq C\psi(\rho)\rho^{-\bar{\alpha}}$ for all $\rho \geq 1$. Since $I_\psi \subset (0,\bar{\alpha})$, by taking limit $\rho \to +\infty$, we arrive at $[u]_{\varphi+\bar{\alpha};\mathbb{R}^n} = 0$ which conclude that u is a polynomial of degree $d(= \lfloor m_{\varphi\psi} \rfloor = \lfloor m_{\varphi+\bar{\alpha}} \rfloor)$. □

4.3 Evans–Krylov-Type Estimates

We prove Theorem 4.0.1 in this section utilizing the Liouville-type theorem. The following proposition is the key ingredient in the proof of Theorem 4.0.1, and it will also be used for the Schauder-type estimates in the next section.

Proposition 4.3.1. *Let $\bar{\alpha}$ be the constant in Theorem 4.2.1, assume* (4.0.5), *and let $\alpha \in (0, m_\psi)$ satisfy* (4.2.1). *Let $\mathcal{I}$ be a concave translation invariant operator which is elliptic with respect to $\mathcal{L}_0(\varphi)$, and suppose that $f \in C^\psi(\overline{B_1})$ and $v \in C^{\varphi\psi}(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$ satisfy*

$$[f]_{\psi;B_1} + \sup_{L \in \mathcal{L}_0(\varphi)} [Lv]_{\psi;B_1} \leq C_0.$$

If $u \in C^{\varphi+\alpha}(\mathbb{R}^n)$ solves

$$\mathcal{I}(u+v) = f \quad \text{in } B_1,$$

then $u \in C^{\varphi\psi}(\overline{B_{1/2}})$ and

$$[u]_{\varphi\psi;B_{1/2}} \leq C \left(\|u\|_{\varphi+\alpha;\mathbb{R}^n} + C_0 \right),$$

where C is a universal constant depending only on n, λ, Λ, a, σ_0, ψ, and $m_{\varphi\psi} - \lfloor m_{\varphi\psi} \rfloor$.

Proof. Assume to the contrary that for each integer $k \geq 0$, there exist u_k, v_k, f_k, and $\mathcal{I}_k$, such that $\mathcal{I}_k(u_k + v_k) = f_k$ in B_1 and

$$[u_k]_{\varphi\psi;B_{1/2}} > k \left(\|u_k\|_{\varphi+\alpha;\mathbb{R}^n} + C_{0,k} \right),$$

where $C_{0,k} = [f_k]_{\psi;B_1} + \sup_{L \in \mathcal{L}_0(\varphi)} [Lv_k]_{\psi;B_1}$. We may always assume that

$$\|u_k\|_{\varphi+\alpha;\mathbb{R}^n} + C_{0,k} = 1 \quad \text{and} \quad [u_k]_{\varphi\psi;B_{1/2}} > k, \tag{4.3.1}$$

by considering $\bar{u}_k := K^{-1}u_k$, $\bar{v}_k := K^{-1}v_k$, $\bar{f}_k := K^{-1}f_k$, and $\bar{\mathcal{I}}_k u := K^{-1}\mathcal{I}_k(Ku)$ with $K = \|u_k\|_{\varphi+\alpha;\mathbb{R}^n} + C_{0,k}$, instead of u_k, v_k, f_k, and $\mathcal{I}_k$. Then we have

$$\sup_{k}\sup_{z\in B_{1/2}}\sup_{\rho>0}\frac{\rho^{\alpha}}{\psi(\rho)}[u_k]_{\varphi+\alpha;B_\rho(z)}=+\infty. \tag{4.3.2}$$

Indeed, if the left hand side of (4.3.2) has a finite value, say C_1, then for all $z\in B_{1/2}$ and for all $\rho\in(0,1)$, we obtain

$$\begin{aligned}\|D^d u_k(z+\cdot)-D^d u_k(z)\|_{L^\infty(B_\rho)}&\le[u_k]_{\varphi+\alpha;B_\rho(z)}\varphi(\rho)\rho^{\alpha-d}\\&\le C_1\varphi(\rho)\psi(\rho)\rho^{-d},\end{aligned} \tag{4.3.3}$$

where $d:=\lfloor m_{\varphi+\bar\alpha}\rfloor=\lfloor m_{\varphi+\alpha}\rfloor=\lfloor m_{\varphi\psi}\rfloor$. Since (4.3.3) contradicts to (4.3.1) for k sufficiently large, the claim (4.3.2) holds.

We define the function

$$\theta(\rho):=\sup_{k}\sup_{z\in B_{1/2}}\sup_{\rho'>\rho}\frac{(\rho')^{\alpha}}{\psi(\rho')}[u_k]_{\varphi+\alpha;B(z,\rho')},$$

which is monotone non-increasing by definition and satisfies $\lim_{\rho\to0}\theta(\rho)=+\infty$ by (4.3.2). Moreover, we know from $\|u_k\|_{\varphi+\alpha;\mathbb{R}^n}\le1$ and the assumption (4.2.1) that $\theta(\rho)<+\infty$ for $\rho>0$. For every positive integer j, there are $\rho_j\ge1/j$, k_j, and $z_j\in B_{1/2}$ such that

$$\frac{1}{2}\theta(\rho_j)\le\frac{1}{2}\theta(1/j)\le\frac{\rho_j^{\alpha}}{\psi(\rho_j)}[u_{k_j}]_{\varphi+\alpha;B(z_j,\rho_j)}\le\theta(1/j). \tag{4.3.4}$$

Note that we have $\rho_j\to0$ as $j\to+\infty$ since otherwise $\frac{\rho_j^{\alpha}}{\psi(\rho_j)}[u_{k_j}]_{\varphi+\alpha;B(z_j,\rho_j)}$ stays bounded while $\theta(1/j)$ blows up as $j\to+\infty$. We define p_j by

$$p_j:=\arg\min_{p\in\mathcal{P}_d}\int_{B(z_j,\rho_j)}(u_{k_j}(x)-p_j(x-z))^2\,\mathrm{d}x, \tag{4.3.5}$$

where $\mathcal{P}_d$ denotes the linear space of polynomials whose degrees are at most d. In other words, the polynomial p_j best fits u_{k_j} in $B(z_j,\rho_j)$ by least squares. Let us consider a blow up sequence

$$\bar u_j(\bar x):=\frac{u_{k_j}(z_j+\rho_j\bar x)-p_j(\rho_j\bar x)}{\varphi(\rho_j)\psi(\rho_j)\theta(\rho_j)}.$$

Then it follows from (4.3.5) that for all $j \geq 1$,

$$\int_{B_1} \bar{u}_j(x) q(x)\, \mathrm{d}x = 0 \quad \text{for all } q \in \mathcal{P}_d, \tag{4.3.6}$$

which is the optimality condition for the least squares. By Lemma 4.1.3 and (4.3.4), we obtain for $\bar{\varphi}_j(r) = \varphi(\rho_j r)/\varphi(\rho_j)$,

$$\begin{aligned}
[\bar{u}_j]_{\bar{\varphi}_j+\alpha;B_1} &= \frac{\rho_j^\alpha}{\psi(\rho_j)\theta(\rho_j)} \left[\frac{u_{k_j}(z_j+\rho_j\cdot) - p_j(\rho_j\cdot)}{\varphi(\rho_j)\rho_j^\alpha} \right]_{\bar{\varphi}_j+\alpha;B_1} \\
&= \frac{\rho_j^\alpha}{\psi(\rho_j)\theta(\rho_j)} [u_{k_j}]_{\varphi+\alpha;B(z_j,\rho_j)} \geq \frac{1}{2},
\end{aligned} \tag{4.3.7}$$

where we used the fact that $d = \lfloor m_{\bar{\varphi}_j+\alpha} \rfloor < m_{\bar{\varphi}_j+\alpha}$ and that $[p_j(\rho_j\cdot)]_{\bar{\varphi}_j+\alpha;B_1} = 0$. The assumptions (4.0.5) and (4.2.1) are used here.

Recall that Lemma 4.1.2 shows that there is a subsequence of $\bar{\varphi}_j$, that we call again $\bar{\varphi}_j$, converging locally uniformly to some function $\bar{\varphi}$. We will prove that there is a subsequence of $\bar{u}_j$ converging in $C^{m_\varphi+\alpha-\varepsilon}_{\mathrm{loc}}(\mathbb{R}^n)$ to a function $\bar{u} \in C^{\bar{\varphi}+\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$ for some $\varepsilon > 0$ small, and that $\bar{u}$ satisfies all the assumptions in Theorem 4.2.1 with φ and ψ replaced by $\bar{\varphi}$ and r^{M_ψ}, respectively. For this, let us prove the following uniform estimates of $\bar{u}_j$:

$$\|\bar{u}_j\|'_{\bar{\varphi}_j+\alpha;B_R} \leq C\bar{\varphi}_j(R)R^{M_\psi} \quad \text{for all } R \geq 1. \tag{4.3.8}$$

We use Lemma 4.1.3, the fact that $[p_j(\rho_j\cdot)]_{\bar{\varphi}_j+\alpha;B_R} = 0$, and the monotonicity of θ, to obtain

$$\begin{aligned}
[\bar{u}_j]_{\bar{\varphi}_j+\alpha;B_R} &= \frac{\rho_j^\alpha}{\psi(\rho_j)\theta(\rho_j)} [u_{k_j}]_{\varphi+\alpha;B(z_j,\rho_jR)} \\
&\leq \frac{\rho_j^\alpha}{\psi(\rho_j)\theta(\rho_j)} \frac{\psi(\rho_jR)\theta(\rho_jR)}{(\rho_jR)^\alpha} \leq \frac{\psi(\rho_jR)}{\psi(\rho_j)} R^{-\alpha}.
\end{aligned}$$

By the definition of M_ψ, we see that $\psi(\rho_jR)/\psi(\rho_j) \leq CR^{M_\psi}$. Thus, we obtain

$$[\bar{u}_j]'_{\bar{\varphi}_j+\alpha;B_R} = \bar{\varphi}_j(R)R^\alpha [\bar{u}_j]_{\bar{\varphi}_j+\alpha;B_R} \leq C\bar{\varphi}_j(R)R^{M_\psi}. \tag{4.3.9}$$

Moreover, (4.3.9) with $R = 1$ implies that $\|\bar{u}_j - p\|_{L^\infty(B_1)} \le C$ for some $p \in \mathcal{P}_d$. Then (4.3.6) gives $\|\bar{u}_j\|_{L^\infty(B_1)} \le C$. Therefore, by using the interpolation inequality, the uniform estimates (4.3.8) is obtained, as in [74].

Since $m_{\bar{\varphi}_j+\alpha} = m_{\varphi+\alpha} \notin \mathbb{N}$, we have an inclusion $C^{\bar{\varphi}_j+\alpha}(\overline{B_R}) \subset C^{m_\varphi+\alpha}(\overline{B_R})$ with

$$\|u\|_{m_\varphi+\alpha;B_R} \le C(a,R)\|u\|_{\bar{\varphi}_j+\alpha;B_R}. \tag{4.3.10}$$

Indeed, since

$$\bar{\varphi}_j(|x-y|)|x-y|^\alpha \le a\bar{\varphi}_j(2R)\left(\frac{|x-y|}{2R}\right)^{\sigma_1}|x-y|^\alpha \le a^2(2R)^{\sigma_2-\sigma_1}|x-y|^{m_\varphi+\alpha},$$

we have

$$\begin{aligned}
[u]_{m_\varphi+\alpha;B_R} &= \sup_{x,y\in B_R}\frac{|D^d u(x) - D^d u(y)|}{|x-y|^{m_\varphi+\alpha-2}} \\
&\le a(2R)^2 \sup_{x,y\in B_R}\frac{|D^d u(x) - D^d u(y)|}{\bar{\varphi}_j(|x-y|)|x-y|^{\alpha-2}} = a(2R)^2[u]_{\bar{\varphi}_j+\alpha;B_R}.
\end{aligned}$$

Thus, (4.3.8) and (4.3.10) shows that for each $R \ge 1$, we find $C > 0$ such that $\|\bar{u}_j\|'_{m_\varphi+\alpha;B_R} \le C$. Therefore, the Arzelà-Ascoli theorem and the standard diagonal sequence argument yields that there is a subsequence of $\bar{u}_j$ converging locally in $C^{m_\varphi+\alpha-\varepsilon}(\mathbb{R}^n)$ to some function $\bar{u} \in C^{\bar{\varphi}+\alpha}_{\mathrm{loc}}(\mathbb{R}^n)$, where ε is a small constant such that $d < m_\varphi + \alpha - \varepsilon$.

Let us next check that $\bar{u}$ satisfies all the assumptions (i), (ii), and (iii), in Theorem 4.2.1, with φ and ψ replaced by $\bar{\varphi}$ and r^{M_ψ}, respectively. The assumption (i) is obtained by simply passing to the limit (4.3.8). In order to check (iii), we set

$$\bar{w}_j(\bar{x}) := u_{k_j}(z_j + \rho_j\bar{x}) + v_{k_j}(z_j + \rho_j\bar{x}).$$

We know from Proposition 4.1.1 that the function $\bar{w}_j$ satisfies

$$\bar{\mathcal{I}}_j\bar{w}_j = \bar{f}_j \quad \text{in } B(-z_j/\rho_j, 1/\rho_j), \tag{4.3.11}$$

where

$$\begin{aligned}\bar{\mathcal{I}}_j w(\bar{x}) &:= \varphi(\rho_j)\frac{c_{\bar{\varphi}_j}}{c_\varphi}\mathcal{I}_{k_j}(w((\cdot - z_j)/\rho_j))(z_j + \rho_j\bar{x}) \quad \text{and} \\ \bar{f}_j(\bar{x}) &:= \varphi(\rho_j)\frac{c_{\bar{\varphi}_j}}{c_\varphi} f_{k_j}(z_j + \rho_j\bar{x}),\end{aligned}$$

and that $\bar{\mathcal{I}}_j$ is elliptic with respect to $\mathcal{L}_0(\bar{\varphi}_j)$. Let μ be a nonnegative $L^1(\mathbb{R}^n)$ function with compact support. Then we have

$$\mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)}\left(\fint \bar{w}_j(\cdot + \bar{h})\,\mathrm{d}\mu(\bar{h}) - \bar{w}_j\right) \geq \bar{\mathcal{I}}_j\left(\fint \bar{w}_j(\cdot + \bar{h})\,\mathrm{d}\mu(\bar{h})\right) - \bar{\mathcal{I}}_j\bar{w}_j. \tag{4.3.12}$$

Since $\mathcal{I}$ is concave and translation invariant, so is $\bar{\mathcal{I}}_j$, and hence

$$\begin{aligned}\bar{\mathcal{I}}_j\left(\fint \bar{w}_j(\cdot + \bar{h})\,\mathrm{d}\mu(\bar{h})\right)(\bar{x}) &\geq \fint \bar{\mathcal{I}}_j(\bar{w}_j(\cdot + \bar{h}))(\bar{x})\,\mathrm{d}\mu(\bar{h}) \\ &= \fint \bar{\mathcal{I}}_j\bar{w}_j(\bar{x} + \bar{h})\,\mathrm{d}\mu(\bar{h}).\end{aligned} \tag{4.3.13}$$

We take j sufficiently large so that $\operatorname{supp}\mu \subset B(-z_j/\rho_j, 1/\rho_j)$, then by (4.3.11) and (4.3.1), we obtain

$$\begin{aligned}\fint \bar{\mathcal{I}}_j\bar{w}_j(\bar{x} + \bar{h})\,\mathrm{d}\mu(\bar{h}) - \bar{\mathcal{I}}_j\bar{w}_j(\bar{x}) &= \fint \left(\bar{f}_j(\bar{x} + \bar{h}) - \bar{f}_j(\bar{x})\right)\mathrm{d}\mu(\bar{h}) \\ &\geq -\varphi(\rho_j)\frac{c_{\bar{\varphi}_j}}{c_\varphi}\fint \psi(|\rho_j\bar{h}|)\,\mathrm{d}\mu(\bar{h}).\end{aligned} \tag{4.3.14}$$

Combining (4.3.12), (4.3.13), and (4.3.14), we have

$$-\varphi(\rho_j)\frac{c_{\bar{\varphi}_j}}{c_\varphi}\fint \psi(|\rho_j\bar{h}|)\,\mathrm{d}\mu(\bar{h}) \leq \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)}\left(\fint \bar{w}_j(\cdot + \bar{h})\,\mathrm{d}\mu(\bar{h}) - \bar{w}_j\right)(\bar{x}).$$

Set

$$\bar{v}_j(\bar{x}) := \frac{v_{k_j}(z_j + \rho_j\bar{x})}{\varphi(\rho_j)\psi(\rho_j)\theta(\rho_j)},$$

then we have $\bar{w}_j = \varphi(\rho_j)\psi(\rho_j)\theta(\rho_j)(\bar{u}_j + \bar{v}_j) + p_j(\rho_j \cdot)$. Since $d \leq 2$, we obtain

$$\delta(p_j(\rho_j \cdot), \bar{x} + \bar{h}, \bar{y}) - \delta(p_j(\rho_j \cdot), \bar{x}, \bar{y}) = 0$$

for all $\bar{x}, \bar{y}, \bar{h} \in \mathbb{R}^n$, and hence,

$$\begin{aligned}
&- \frac{1}{\theta(\rho_j)} \frac{c_{\bar{\varphi}_j}}{c_\varphi} \fint \bar{\psi}_j(|\bar{h}|) \,\mathrm{d}\mu(\bar{h}) \\
&\leq \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)} \left(\fint (\bar{u}_j + \bar{v}_j)(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - (\bar{u}_j + \bar{v}_j) \right) \\
&\leq \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)} \left(\fint \bar{u}_j(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - \bar{u}_j \right) + \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)} \left(\fint \bar{v}_j(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - \bar{v}_j \right)
\end{aligned}$$

in $B(-z_j/\rho_j, 1/\rho_j)$. Using (4.1.2) and (4.3.1), we have for $x = z_j + \rho_j \bar{x} \in B_1$,

$$\begin{aligned}
&\mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)} \left(\fint \bar{v}_j(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - \bar{v}_j \right)(\bar{x}) \\
&= \frac{1}{\psi(\rho_j)\theta(\rho_j)} \frac{c_{\bar{\varphi}_j}}{c_\varphi} \mathcal{M}^+_{\mathcal{L}_0(\varphi)} \left(\fint \left(v_{k_j}(\cdot + h) - v_{k_j} \right) \,\mathrm{d}\mu\left(\frac{h}{\rho_j} \right) \right)(x) \\
&\leq \frac{1}{\theta(\rho_j)} \frac{c_{\bar{\varphi}_j}}{c_\varphi} \fint \bar{\psi}_j(|\bar{h}|) \,\mathrm{d}\mu(\bar{h}).
\end{aligned} \tag{4.3.15}$$

Therefore, we obtain

$$- \frac{1}{\theta(\rho_j)} \frac{c_{\bar{\varphi}_j}}{c_\varphi} \fint \bar{\psi}_j(|\bar{h}|) \,\mathrm{d}\mu(\bar{h}) \leq \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi}_j)} \left(\fint \bar{u}_j(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - \bar{u}_j \right)(\bar{x}). \tag{4.3.16}$$

Since $\theta(\rho_j) \to +\infty$, $c_{\bar{\varphi}_j} \to c_\varphi$, and $\fint \bar{\psi}_j(|\bar{h}|) \,\mathrm{d}\mu(\bar{h}) \to \fint \bar{\psi}(|\bar{h}|) \,\mathrm{d}\mu(\bar{h})$ as $j \to +\infty$, the left hand side of (4.3.16) converges to 0 as $j \to +\infty$. For the right hand side of (4.3.16), we use the dominated convergence theorem to pass to the limit, which is guaranteed by the growth control (4.3.8). Therefore, we arrive at

$$0 \leq \mathcal{M}^+_{\mathcal{L}_0(\bar{\varphi})} \left(\fint \bar{u}(\cdot + \bar{h}) \,\mathrm{d}\mu(\bar{h}) - u \right) \quad \text{in } \mathbb{R}^n,$$

which gives the assumption (iii). The assumption (ii) is obtained in a similar way.

Since we have checked all the assumptions of Theorem 4.2.1, we conclude that $\bar{u}$ is a polynomial of degree $\lfloor m_{\bar{\varphi}} + M_\psi \rfloor$, which is equal to d by the assumption (4.0.5). Passing (4.3.6) to the limit, we see that $\bar{u}$ is orthogonal to every polynomial of degree d in B_1. This shows that $\bar{u}$ must be zero, which contradicts to the limit of (4.3.7). Therefore, we obtain the desired result. $\square$

We are now ready to prove Theorem 4.0.1 by using Proposition 4.3.1.

Proof of Theorem 4.0.1. Let $z \in B_1$ and $\rho \in (0,1)$ be such that $B_\rho(z) \subset B_1$. By Proposition 4.1.1, the rescaled function $\bar{u}(\bar{x}) := u(z + \rho\bar{x})$ solves the equation $\bar{\mathcal{I}}\bar{u} = \bar{f}$ in B_1, where

$$\bar{\mathcal{I}}\bar{u}(\bar{x}) := \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}\mathcal{I}(\bar{u}((\cdot - z)/\rho))(z + \rho\bar{x})$$

is elliptic with respect to $\mathcal{L}_0(\bar{\varphi})$ and $\bar{f}(\bar{x}) := \varphi(\rho)\frac{c_{\bar{\varphi}}}{c_\varphi}f(z + \rho\bar{x})$. Let η be a cut-off function such that $\eta = 1$ in $B_{3/4}$ and $\eta = 0$ outside B_1. Since the function $\eta\bar{u} \in C^{\varphi+\alpha}(\mathbb{R}^n)$ solves

$$\bar{\mathcal{I}}(\eta\bar{u} + \bar{v}) = \bar{f} \quad \text{in } B_{1/2},$$

where $\bar{v} = (1-\eta)\bar{u}$, we have

$$[\eta\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \le C\left(\|\eta\bar{u}\|_{\bar{\varphi}+\alpha;\mathbb{R}^n} + [\bar{f}]_{\bar{\psi};B_{1/2}} + \sup_{L\in\mathcal{L}_0(\bar{\varphi})}[L\bar{v}]_{\bar{\psi};B_{1/2}}\right)$$

by Proposition 4.3.1. We first claim that

$$\sup_{L\in\mathcal{L}_0(\bar{\varphi})}[L\bar{v}]_{\bar{\psi};B_{1/2}} \le C[\bar{u}]_{\bar{\psi};\mathbb{R}^n}. \tag{4.3.17}$$

Since $\bar{v} = (1-\eta)\bar{u} = 0$ in $B_{3/4}$, we have for $\bar{x}, \bar{x}' \in B_{1/2}$,

$$\begin{aligned} |L\bar{v}(\bar{x}) - L\bar{v}(\bar{x}')| &\leq \int_{\mathbb{R}^n \setminus B_{1/4}} |\delta(\bar{v}, \bar{x}, \bar{y}) - \delta(\bar{v}, \bar{x}', \bar{y})| \frac{c_{\bar{\varphi}}}{|\bar{y}|^n \bar{\varphi}(|\bar{y}|)} \, \mathrm{d}\bar{y} \\ &\leq 4[\bar{v}]_{\bar{\psi};\mathbb{R}^n} \bar{\psi}(|\bar{x} - \bar{x}'|) \int_{\mathbb{R}^n \setminus B_{1/4}} \frac{c_{\bar{\varphi}}}{|\bar{y}|^n \bar{\varphi}(|\bar{y}|)} \, \mathrm{d}\bar{y}. \end{aligned}$$

We use (2.1.3) to obtain that

$$\int_{\mathbb{R}^n \setminus B_{1/4}} \frac{c_{\bar{\varphi}}}{|\bar{y}|^n \bar{\varphi}(|\bar{y}|)} d\bar{y} \leq C(2-\sigma_1) \int_{1/4}^{\infty} r^{-1-\sigma_1} dr \leq C(n, a, \sigma_0),$$

which gives rise to $|L\bar{v}(\bar{x}) - L\bar{v}(\bar{x}')| \leq C[\bar{u}]_{\bar{\psi};\mathbb{R}^n} \bar{\psi}(|\bar{x} - \bar{x}'|)$. Thus, the claim (4.3.17) is proved, and hence

$$[\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \leq C \left(\|\bar{u}\|_{\bar{\varphi}+\alpha;B_1} + [\bar{u}]_{\bar{\psi};\mathbb{R}^n} + [\bar{f}]_{\bar{\psi};B_{1/2}} \right).$$

Scaling back, we obtain

$$[u]'_{\varphi\psi;B_{\rho/4}(z)} \leq C \left(\|u\|'_{\varphi+\alpha;B_\rho(z)} + \psi(\rho)[u]_{\psi;\mathbb{R}^n} + \varphi(\rho) \frac{c_{\bar{\varphi}}}{c_\varphi} [f]'_{\psi;B_{\rho/2}(z)} \right).$$

We know from Lemma 2.1.2 and $\rho < 1$ that

$$\varphi(\rho) \frac{c_{\bar{\varphi}}}{c_\varphi} = \rho^2 \frac{\underline{C}_\varphi(1)}{\underline{C}_\varphi(\rho)} \leq \rho^2 \left(1 + a^2 \rho^{-(2-\sigma_1)}\right) \leq C. \tag{4.3.18}$$

Moreover, since $\rho < 1$ we have $\psi(\rho) \leq C$. Thus, with the help of the interpolation inequality with a small constant $\varepsilon > 0$, we have

$$\|u\|'_{\varphi\psi;B_{\rho/4}(z)} \leq C \left(\varepsilon \|u\|'_{\varphi\psi;B_\rho(z)} + \|u\|_{\psi;\mathbb{R}^n} + \|f\|'_{\psi;B_{\rho/2}(z)} \right). \tag{4.3.19}$$

Note that (4.3.19) can be written, by using the interior norms, as

$$\|u\|^*_{\varphi\psi;B_1} \leq C \left(\varepsilon \|u\|^*_{\varphi\psi;B_1} + \|u\|_{\psi;\mathbb{R}^n} + \|f\|^*_{\psi;B_1} \right).$$

By taking ε sufficiently small so that $C\varepsilon \leq 1/2$, we arrive at

$$\|u\|^*_{\varphi\psi;B_1} \leq C\left(\|u\|_{\psi;\mathbb{R}^n} + \|f\|^*_{\psi;B_1}\right),$$

which concludes the theorem. □

4.4 Schauder-Type Estimates

In this section we establish the Schauder-type estimates for non-translation invariant fully nonlinear equations. Both Theorem 4.0.2 and Theorem 4.0.3 will follow from the following intermediate statement that shows the "freezing coefficients" step.

Proposition 4.4.1. *Let $\bar{\alpha}$ be the constant in Theorem 4.0.1, assume* (4.0.5), *and let $\alpha \in (0, m_\psi)$ satisfy* (4.2.1). *Let $\mathcal{L}$ be either $\mathcal{L}_0(\varphi)$ or $\mathcal{L}_\psi(\varphi)$, and let $\mathcal{I}$ be a non-translation invariant operator which is elliptic with respect to $\mathcal{L}$. Suppose that*

$$\beta_{\mathcal{I}-\mathcal{I}_0}(x, x') \leq A_0\psi(|x - x'|) \quad \text{for all } x, x' \in B_1(0), \tag{4.4.1}$$

with $A_0 \leq 1$, and that $\mathcal{I}_0$ satisfies the Evans–Krylov-type estimates for $B_1(0)$. If $u \in C^{\varphi\psi}(\overline{B_1}) \cap C^{\varphi+\alpha}(\mathbb{R}^n)$ solves, for $f \in C^\psi(\overline{B_1})$ and $v \in C^{\varphi\psi}(\overline{B_1}) \cap L^\infty(\mathbb{R}^n)$,

$$\mathcal{I}(u + v, x) = f(x) \quad \text{in } B_1,$$

then

$$[u]_{\varphi\psi;B_{1/2}} \\ \leq C\left(\|u\|_{\varphi+\alpha;\mathbb{R}^n} + A_0\|u + v\|_{\varphi\psi;B_1} + \|u + v\|_\infty + [f]_{\psi;B_1} + \sup_{L\in\mathcal{L}}[Lv]_{\psi;B_1}\right),$$

where C is a universal constant depending only on n, λ, Λ, a, σ_0, ψ, and $m_{\varphi\psi} - \lfloor m_{\varphi\psi}\rfloor$.

If $\mathcal{I}$ is concave and elliptic with respect to $\mathcal{L}_0(\varphi)$, then the assumption that $\mathcal{I}_0$ satisfies the Evans–Krylov-type estimates is fulfilled by Proposition 4.3.1. Note that this is also true if $\mathcal{I}$ is concave and elliptic with respect to $\mathcal{L}_\psi(\varphi)$ since Proposition 4.3.1 holds true with $\sup_{L\in\mathcal{L}_0(\varphi)}[Lv]_{\psi;B_1}$ replaced by $\sup_{L\in\mathcal{L}_\psi(\varphi)}[Lv]_{\psi;B_1}$. It is easily checked by observing that the inequality (4.3.15) is the only part where $\sup_{L\in\mathcal{L}_0(\varphi)}[Lv]_{\psi;B_1}$ is used throughout the proof. Thus, concave non-translation invariant operators (elliptic with respect to $\mathcal{L}_0(\varphi)$ or $\mathcal{L}_\psi(\varphi)$) are examples of operators for Proposition 4.4.1, and hence for Theorem 4.0.2 or Theorem 4.0.3, respectively.

Proof. We write the equation $\mathcal{I}(u+v,x) = f(x)$ by

$$\mathcal{I}_0(u+v)(x) = f(x) - (\mathcal{I}(u+v,x) - \mathcal{I}_0(u+v)(x)) \quad \text{in } B_1,$$

Since $\mathcal{I}_0$ satisfies the Evans–Krylov-type estimates in B_1, we have

$$\begin{aligned}&[u]_{\varphi\psi;B_{1/2}}\\ &\leq C\left(\|u\|_{\varphi+\alpha;\mathbb{R}^n} + [f]_{\psi;B_1} + [\mathcal{I}(u+v,\cdot) - \mathcal{I}_0(u+v)]_{\psi;B_1} + \sup_{L\in\mathcal{L}(\varphi)}[Lv]_{\psi;B_1}\right).\end{aligned} \tag{4.4.2}$$

Thus, it only remains to estimate $[\mathcal{I}(u+v,\cdot) - \mathcal{I}_0(u+v)]_{\psi;B_1}$. But, the assumption (4.4.1) shows that

$$\begin{aligned}&[\mathcal{I}(u+v,\cdot) - \mathcal{I}_0(u+v)]_{\psi;B_1}\\ &\leq \sup_{x,x'\in B_1}\frac{\beta_{\mathcal{I}-\mathcal{I}_0}(x,x')}{\psi(|x-x'|)}\left(\|u+v\|'_{\varphi\psi;B_1} + \|u+v\|_{L^\infty(\mathbb{R}^n)}\right) \qquad (4.4.3)\\ &\leq A_0\left(\|u+v\|_{\varphi\psi;B_1} + \|u+v\|_{L^\infty(\mathbb{R}^n)}\right).\end{aligned}$$

Therefore, the result follows from (4.4.2) and (4.4.3) since $A_0 \leq 1$. $\square$

We point out that the proofs of Theorem 4.0.2 and Theorem 4.0.3 differ only in the control of quantity $\sup_{L\in\mathcal{L}}[Lv]_{\psi;B_1}$ in Proposition 4.4.1. It can be controlled by using the C^ψ Hölder regularity of solutions u in Theorem 4.0.2. However, if both solutions and kernels are not regular enough, then this quantity cannot be controlled (see Section 4.5). This is why we require some regularity of kernels when solutions are merely bounded. Let us first prove Theorem 4.0.2.

Proof of Theorem 4.0.2. As in the proof of Theorem 4.0.1, let $B_{\rho/2}(z) \subset B_\rho(z) \subset B_1$ and consider the rescaled equation

$$\bar{\mathcal{I}}(\bar{u}, \bar{x}) = \bar{f}(\bar{x}) \quad \text{in } B_{1/2}, \tag{4.4.4}$$

where $\bar{u}$, $\bar{\mathcal{I}}$, and $\bar{f}$ are given as Proposition 4.1.1, with $\bar{\varphi}(r) := \varphi(\rho r)/\varphi(\rho)$ and $\bar{\psi}(r) := \psi(\rho r)/\psi(\rho)$. Then the assumption (4.0.6) reads as

$$\beta_{\bar{\mathcal{I}}-\bar{\mathcal{I}}_0}(\bar{x}, \bar{x}') \le A_0 \psi(\rho) \bar{\psi}(|\bar{x} - \bar{x}'|) \quad \text{for all } \bar{x}, \bar{x}' \in B_{1/2}.$$

Indeed, for $\bar{x}, \bar{x}' \in B_{1/2}$ and $\bar{w} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_{1/2}}) \cap L^\infty(\mathbb{R}^n)$, let $x = z + \rho \bar{x}$, $x' = z + \rho \bar{x}'$, and $w(x) = \bar{w}((x-z)/\rho)$. Then by Lemma 4.1.4 and (4.3.18), we have

$$\begin{aligned}
\beta_{\bar{\mathcal{I}}-\bar{\mathcal{I}}_0}(\bar{x}, \bar{x}') &= \sup_{\bar{w}} \frac{|\bar{\mathcal{I}}(\bar{w}, \bar{x}) - \bar{\mathcal{I}}_0 \bar{w}(\bar{x}) - (\bar{\mathcal{I}}(\bar{w}, \bar{x}') - \bar{\mathcal{I}}_0 \bar{w}(\bar{x}'))|}{\|\bar{w}\|'_{\bar{\varphi}\bar{\psi}; B_{1/2}} + \|\bar{w}\|_{L^\infty(\mathbb{R}^n)}} \\
&= \varphi(\rho) \frac{c_{\bar{\varphi}}}{c_\varphi} \sup_{w} \frac{|\mathcal{I}(w, x) - \mathcal{I}_z w(x) - (\mathcal{I}(w, x') - \mathcal{I}_z w(x'))|}{\|w\|'_{\varphi\psi; B_{\rho/2}(z)} + \|w\|_{L^\infty(\mathbb{R}^n)}} \\
&\le \beta_{\mathcal{I}-\mathcal{I}_z}(x, x') \le A_0 \psi(|x - x'|) = A_0 \psi(\rho) \bar{\psi}(|\bar{x} - \bar{x}'|).
\end{aligned}$$

Furthermore, $\bar{\mathcal{I}}_0$ has Evans–Krylov-type estimate in $B_{1/2}$ since $\mathcal{I}_z$ has it in $B_{\rho/2}(z)$. To apply Proposition 4.4.1, we make $\bar{A}_0 := A_0 \psi(\rho) \le \varepsilon_0 \le 1$ by taking $\rho = \rho(A_0, \psi) > 0$ sufficiently small. The universal constant ε_0 will be chosen later. Let η be a cut-off function supported in B_1 satisfying $\eta \equiv 1$ on $B_{3/4}$ and write the equation (4.4.4) as $\bar{\mathcal{I}}(\eta \bar{u} + \bar{v}, \bar{x}) = \bar{f}(\bar{x})$ with $\bar{v} = (1 - \eta)\bar{u}$. Since $u \in C^{\varphi\psi}(B_1) \cap C^\psi(\mathbb{R}^n)$, we have $\bar{u} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_1}) \cap C^{\bar{\psi}}(\mathbb{R}^n)$, and hence $\eta \bar{u} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_{1/2}}) \cap C^{\bar{\varphi}+\alpha}(\mathbb{R}^n)$ and $\bar{v} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_{1/2}}) \cap C^{\bar{\psi}}(\mathbb{R}^n)$. Thus, we obtain by the rescaled version of Proposition 4.4.1,

$$[\eta\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \\ \leq C\left(\|\eta\bar{u}\|_{\bar{\varphi}+\alpha;\mathbb{R}^n} + \bar{A}_0\|\bar{u}\|_{\bar{\varphi}\bar{\psi};B_{1/2}} + \|\bar{u}\|_\infty + [\bar{f}]_{\bar{\psi};B_{1/2}} + \sup_{L\in\mathcal{L}_0(\bar{\varphi})}[L\bar{v}]_{\bar{\psi};B_{1/2}}\right).$$

Notice that it follows from the computation (4.3.17) that

$$[\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \leq C\left(\|\bar{u}\|_{\bar{\varphi}+\alpha;B_1} + \varepsilon_0\|\bar{u}\|_{\bar{\varphi}\bar{\psi};B_{1/2}} + \|\bar{u}\|_{\bar{\psi};\mathbb{R}^n} + [\bar{f}]_{\bar{\psi};B_{1/2}}\right).$$

By scaling back and then using (4.3.18) and $\psi(\rho) \leq C$, we have

$$[u]'_{\varphi\psi;B_{\rho/4}(z)} \leq C\left(\|u\|'_{\varphi+\alpha;B_\rho(z)} + \varepsilon_0\|u\|'_{\varphi\psi;B_{\rho/2}(z)} + \|u\|_{\psi;\mathbb{R}^n} + [f]'_{\psi;B_{\rho/2}(z)}\right).$$

Using the interpolation inequalities, we obtain that

$$\|u\|'_{\varphi\psi;B_{\rho/4}(z)} \leq C\left(2\varepsilon_0\|u\|'_{\varphi\psi;B_{\rho/2}(z)} + \|u\|_{\psi;\mathbb{R}^n} + \|f\|'_{\psi;B_{\rho/2}(z)}\right),$$

or, in terms of the interior norms, that

$$\|u\|^*_{\varphi\psi;B_1} \leq C\left(2\varepsilon_0\|u\|^*_{\varphi\psi;B_1} + \|u\|_{\psi;\mathbb{R}^n} + \|f\|^*_{\psi;B_1}\right).$$

By taking ε_0 sufficiently small so that $2C\varepsilon_0 \leq 1/2$, we arrive at the desired estimates. $\square$

We finally prove the last theorem. Instead of using the C^ψ Hölder regularity of solutions, we use the regularity of kernels (4.0.8) to estimate the quantity $\sup_{L\in\mathcal{L}_\psi(\varphi)}[Lv]_{\psi;B_1}$.

Proof of Theorem 4.0.3. By the same argument as in Theorem 4.0.2, we have the rescaled equation $\bar{\mathcal{I}}(\eta\bar{u}+\bar{v},\bar{x}) = \bar{f}(\bar{x})$ in $B_{1/2}$. In this case, we have $\eta\bar{u} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_{1/2}}) \cap C^{\bar{\varphi}+\alpha}(\mathbb{R}^n)$ and $\bar{v} \in C^{\bar{\varphi}\bar{\psi}}(\overline{B_{1/2}}) \cap L^\infty(\mathbb{R}^n)$. Thus, we can still apply the rescaled version of Proposition 4.4.1 to obtain

$$[\eta\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \leq C\left(\|\eta\bar{u}\|_{\bar{\varphi}+\alpha;\mathbb{R}^n} + \bar{A}_0\|\bar{u}\|_{\bar{\varphi}\bar{\psi};B_{1/2}} + \|\bar{u}\|_\infty + [\bar{f}]_{\bar{\psi};B_{1/2}} + \sup_{L\in\mathcal{L}_\psi(\varphi)}[L\bar{v}]_{\bar{\psi};B_{1/2}}\right).$$

Let us estimate $\sup_{L\in\mathcal{L}_\psi(\varphi)}[L\bar{v}]_{\bar{\psi};B_{1/2}}$. Since $\bar{v}\equiv 0$ in $B_{3/4}$, we see that for $\bar{x}\in B_{1/2}$ and $\bar{h}\in B_{1/16}$,

$$\begin{aligned}|L\bar{v}(\bar{x}+\bar{h}) - L\bar{v}(\bar{x})| &= \left|2\int_{\mathbb{R}^n}\bar{v}(\bar{x}+\bar{y})(K(\bar{y}-\bar{h})-K(\bar{y}))\,\mathrm{d}\bar{y}\right| \\ &\leq 2\psi(|\bar{h}|)\int_{\mathbb{R}^n\setminus B_{1/8}}|\bar{v}(\bar{x}+\bar{y})|\frac{|K(\bar{y}-\bar{h})-K(\bar{y})|}{\psi(|\bar{h}|)}\,\mathrm{d}\bar{y} \\ &\leq 2\|\bar{v}\|_{L^\infty(\mathbb{R}^n)}\psi(|\bar{h}|)\int_{\mathbb{R}^n\setminus B_{1/8}}\frac{|K(\bar{y}-\bar{h})-K(\bar{y})|}{\psi(|\bar{h}|)}\,\mathrm{d}\bar{y}.\end{aligned}$$

The assumption (4.0.8) implies that

$$\int_{\mathbb{R}^n\setminus B_{\rho_0}}\frac{|K(y)-K(y-h)|}{\psi(|h|)}\,\mathrm{d}y \leq C$$

whenever $|h|\leq\rho_0/2$. Therefore, we obtain

$$|L\bar{v}(\bar{x}+\bar{h}) - L\bar{v}(\bar{x})| \leq C\|\bar{u}\|_{L^\infty(\mathbb{R}^n)}\psi(|\bar{h}|),$$

and hence,

$$[\bar{u}]_{\bar{\varphi}\bar{\psi};B_{1/4}} \leq C\left(\|\bar{u}\|_{\bar{\varphi}+\alpha;B_1} + \varepsilon_0\|\bar{u}\|_{\bar{\varphi}\bar{\psi};B_{1/2}} + \|\bar{u}\|_{L^\infty(\mathbb{R}^n)} + [\bar{f}]_{\bar{\psi};B_{1/2}}\right).$$

As in the proof of Theorem 4.0.2, the standard covering argument finishes the proof. □

4.5 Counterexamples to $C^{\varphi\psi}$ Regularity for Merely Bounded Solutions

In this section we observe that the $C^{\psi}(\mathbb{R}^n)$ assumption on u in Theorem 4.0.1 and Theorem 4.0.2 can not be relaxed to $L^{\infty}(\mathbb{R}^n)$. This can be seen by finding a sequence $\{u_m\}$ of solutions to equations with rough kernels of variable orders in B_1 that satisfy $\|u_m\|_{L^\infty(\mathbb{R}^n)} \leq C$ and $\|u_m\|_{C^{\varphi\psi}(B_{1/2})} \to +\infty$ as $m \to +\infty$, under the assumption (4.0.5). This sequence is given in [74, Section 5] for the case $\varphi = r^\sigma$ and $\psi = r^\alpha$. The construction of the sequence in a general framework is almost the same, but let us show how to construct the sequence for the completeness.

Let us find a sequence in dimension $n = 1$, but this will give the sequence in every dimension by considering rotationally symmetric functions. For every $m \geq 1$, we consider the solution u_m to

$$\begin{cases} \mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m = 0 & \text{in } (-1,1), \\ u_m = 0 & \text{in } [-2,-1] \cup [1,2], \\ u_m = \operatorname{sign} \sin(m\pi x) & \text{in } (-\infty,-2] \cup [2,\infty). \end{cases}$$

For $p > 0$ small enough, the function $\psi(x) = \operatorname{dist}(x, [-1/4, 1/4])^p$ satisfies $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} \psi \leq 0$ in $(-1/4-\varepsilon, -1/4) \cup (1/4, 1/4+\varepsilon)$ for some $\varepsilon > 0$. We use the translation of ψ as a barrier and use Theorem 2.3.6 to obtain $|u_m| \leq C\operatorname{dist}(x, (-\infty,-1] \cup [1,\infty))^p$ in $(-1,1)$. Combining this with Theorem 3.0.3, we have

$$\|u_m\|_{C^\alpha([-2,2])} = \|u_m\|_{C^\alpha([-1,1])} \leq C \tag{4.5.1}$$

for some $\alpha > 0$ small enough and $C > 0$ independent of m.

We next assume that

$$\|u_m\|_{C^{\varphi\psi}((-1/2,1/2))} \leq C, \tag{4.5.2}$$

and find a contradiction by comparing $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m(0)$ and $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m(1/(2m))$, which are 0 by definition. Note that the maximal operator $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}$ can be written by

$$\begin{aligned}\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u(x) &= \int_{\mathbb{R}} \frac{\Lambda\delta_+(u,x,y) - \lambda\delta_-(u,x,y)}{|y|\varphi(|y|)}\,\mathrm{d}y \\ &= \int_{\mathbb{R}} \delta(u,x,y)\frac{\Lambda(\mathrm{sign}(\delta(u,x,y))_+ + \lambda(\mathrm{sign}(\delta(u,x,y)))_-}{|y|\varphi(|y|)}\,\mathrm{d}y,\end{aligned}$$

where we omit the constant c_φ in this section. Since for all $L \in \mathcal{L}_0(\varphi)$ the solution to $Lw = 0$ in $(-1,1)$ with the same boundary data as u_m satisfies $w(0) = 0$ and $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}w \geq Lw = 0$, by the comparison principle we have $u_m(0) \geq 0$. This gives $\delta(u_m, 0, y) = -2u_m(0) \leq 0$ for $|y| > 2$. Thus, we have

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u_m(0) = \int_{\mathbb{R}} \delta(u_m,0,y)\frac{b_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y,$$

where $b_m(y) = \Lambda(\mathrm{sign}(\delta(u_m,0,y)))_+ + \lambda(\mathrm{sign}(\delta(u_m,0,y)))_-$, and $b_m(y) \equiv \lambda$ for $|y| > 2$.

On the other hand, we have

$$\delta\left(u_m, \frac{1}{2m}, y\right) = 2\mathrm{sign}\cos(m\pi y) - 2u_m\left(\frac{1}{2m}\right) \quad \text{for } |y| > 2 + \frac{1}{2m}.$$

Hence, we obtain

$$\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u_m\left(\frac{1}{2m}\right) = \int_{\mathbb{R}} \delta\left(u_m, \frac{1}{2m}, y\right)\frac{\tilde{b}_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y,$$

where

$$\tilde{b}_m(y) = \Lambda\left(\mathrm{sign}\left(\delta\left(u_m, \frac{1}{2m}, y\right)\right)\right)_+ + \lambda\left(\mathrm{sign}\left(\delta\left(u_m, \frac{1}{2m}, y\right)\right)\right)_-,$$

and

$$\tilde{b}_m(y) = \lambda + \frac{\Lambda - \lambda}{2}(1 + \mathrm{sign}\cos(m\pi y)) \quad \text{for } |y| > 2 + \frac{1}{2m}.$$

We know from $|u_m| \leq 1$ in $\mathbb{R}$, $|\{u_m < 0\} \cap (-5,5)| \geq 1$, and $\mathcal{M}^+_{\mathcal{L}_0(\varphi)}u_m = 0$ in $(-1,1)$, that $u_m \leq 1 - \tau$ in $[-1/2, 1/2]$ for some $\tau > 0$ independent of m.

Thus, using $u_m(0) \in [0, 1-\tau]$, we have for all $\gamma \in (0,1)$ and for all m,

$$\int_{|y|>2+\gamma} 2(\operatorname{sign}\cos(m\pi y) - u_m(0)) \frac{\frac{\Lambda-\lambda}{2}(1+\operatorname{sign}\cos(m\pi y))}{|y|\varphi(|y|)}\,\mathrm{d}y \geq 2c_1 > 0,$$

where c_1 is independent of m. For m large enough so that

$$\left| \int_{|y|>2+\gamma} 2\operatorname{sign}\cos(m\pi y)\frac{\lambda}{|y|\varphi(|y|)}\,\mathrm{d}y \right| \leq c_1,$$

we obtain

$$\int_{|y|>2+\gamma} 2(\operatorname{sign}\cos(m\pi y) - u_m(0))\frac{\tilde{b}_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y \geq c_1 - \int_{|y|>2+\gamma} u_m(0)\frac{\lambda}{|y|\varphi(|y|}\,\mathrm{d}y.$$

Therefore, by using (4.5.1), we see that

$$\begin{aligned}
&\int_{|y|>2+\gamma} \delta\left(u_m, \frac{1}{2m}, y\right)\frac{\tilde{b}_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y - \int_{|y|>2+\gamma} \delta\left(u_m, 0, y\right)\frac{b_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y \\
&\geq c_1 - 2\left|u_m\left(\frac{1}{2m}\right) - u_m(0)\right| \int_{|y|>2+\gamma} \frac{\tilde{b}_m(y)}{|y|\varphi(|y|)}\,\mathrm{d}y \geq c_1 - C\left|\frac{1}{2m} - 0\right|^{\alpha}.
\end{aligned} \tag{4.5.3}$$

Let us next prove that

$$\left| \int_{|y|<2-\gamma} \left(\delta\left(u_m, \frac{1}{2m}, y\right)\frac{\tilde{b}_m(y)}{|y|\varphi(|y|)} - \delta\left(u_m, 0, y\right)\frac{b_m(y)}{|y|\varphi(|y|)} \right)\mathrm{d}y \right| \leq C\left(\frac{1}{m}\right)^{\alpha'} \tag{4.5.4}$$

for some $\alpha' \in (0,\alpha)$. By (4.5.2) and (4.5.1), we have for $|y| < 2-\gamma$,

$$\left| \delta(u_m, 0, y) - \delta\left(u_m, \frac{1}{2m}, y\right) \right| \leq C\left(\varphi(|y|)\psi(|y|) \wedge \left|\frac{1}{2m}\right|^{\alpha} \right).$$

By the definition of m_ψ, we have $\psi(|y|) \leq C|y|^{m_\psi}$. Thus, for $\theta \in (0,1)$ and $\alpha' = (1-\theta)\alpha$, we obtain

$$\left| \delta(u_m, 0, y) - \delta\left(u_m, \frac{1}{2m}, y\right) \right| \leq C_1\varphi^{\theta}(|y|)|y|^{\theta m_\psi}\left|\frac{1}{2m}\right|^{\alpha'}. \tag{4.5.5}$$

If we take θ sufficiently close to 1 so that $\theta m_\psi > (1-\theta)M_\varphi$, then we have

$$\int_{|y|<2-\gamma} \frac{\varphi^\theta(|y|)|y|^{\theta m_\psi}}{|y|^n \varphi(|y|)}\,\mathrm{d}y \le \int_{|y|<2-\gamma} \frac{a^{1-\theta}}{\varphi^{1-\theta}(2)} \left|\frac{2}{y}\right|^{(1-\theta)\sigma_2} |y|^{-n+\theta m_\psi}\,\mathrm{d}y \le C. \tag{4.5.6}$$

If $\delta(u_m, 0, y)$ or $\delta(u_m, \frac{1}{2m}, y)$ is greater than $2C_1\varphi^\theta(|y|)|y|^{\theta m_\psi}\left|\frac{1}{2m}\right|^{\alpha'}$, then we have $b = \tilde{b} = \Lambda$. Similarly, if $\delta(u_m, 0, y)$ or $\delta(u_m, \frac{1}{2m}, y)$ is less than $-2C_1\varphi^\theta(|y|)|y|^{\theta m_\psi}\left|\frac{1}{2m}\right|^{\alpha'}$, then we obtain $b = \tilde{b} = \lambda$. In these cases, we use (4.5.5) and (4.5.6) to compute the left hand side of (4.5.4). If both $|\delta(u_m, 0, y)|$ and $|\delta(u_m, 1/(2m), y)|$ are less than $2C_1\varphi^\theta(|y|)|y|^{\theta m_\psi}\left|\frac{1}{2m}\right|^{\alpha'}$, then (4.5.6) is enough to conclude (4.5.4).

Since $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m(0) = 0$ and $\mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m(1/(2m)) = 0$, by (4.5.3) and (4.5.4), we arrive at

$$0 = \mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m\left(\frac{1}{2m}\right) - \mathcal{M}^+_{\mathcal{L}_0(\varphi)} u_m(0) \ge c_1 - C\left|\frac{1}{2m}\right|^{\alpha} - C\left|\frac{1}{2m}\right|^{\alpha'} - C\gamma.$$

By taking $\gamma < c_1/C$ and taking limit $m \to \infty$, we get a contradiction.

Chapter 5

Green Function Estimates

A Green function plays a fundamental role in both potential theory and probability theory, and has numerous applications in many other branches of mathematics. In this chapter we study the Green function for general nonlocal operators in divergence form.

The classical notion of a Green function for differential operators in potential theory have been generalized to deal with second order elliptic operators with rough coefficients. Such operators and the corresponding Green function defined in the weak sense were dealt with by Littman, Stampacchia, and Weinberger [60] for symmetric coefficients, and by Grüter and Widman [42] for non-symmetric coefficients. In [42], the existence, uniqueness, and pointwise upper and lower bounds near the singularity of Green function are investigated via purely analytic methods. In particular, the upper bound for the Green function up to the boundary is also obtained when the domain is sufficiently smooth. The global lower bound of Green function can be found in [81].

Linear differential operators of second order have counterparts in probability theory so that the Green function is given by the transition density (heat kernel) of the corresponding process (see e.g., [8, 33]). The upper and lower bounds of the Dirichlet heat kernel for differential operators or the Brownian motion on bounded $C^{1,1}$ domains were established in [28] and [80], respectively.

For nonlocal operators, Kassmann and Steinhauer [47] extended the notion of Green function defined in the weak sense to nonlocal operators with rough coefficients, and provided the existence result. However, pointwise bounds for Green functions were not obtained at the time because of the lack of the tools for analysis on nonlocal operators.

Instead of the analytic perspective, Green functions for nonlocal operators have been studied through the stochastic interpretation. The first result concerning estimates on Green function for symmetric stable processes was given by Chen and Song [24], and by Kulczycki [57] independently. After these works a number of results about Green function estimates for various processes has been studied. See, for examples, [25, 70, 20, 21, 52, 51, 43], and references therein. The heat kernel and Green function estimates are also obtained via the Dirichlet form theory. See [39, 38].

Such probabilistic methods rely on in many cases the heat kernel estimates, that do not provide the robustness of the estimates. This is observed from the fact that the heat kernels for continuous processes have exponential decays whereas those for jump processes have polynomial decays. Moreover, these methods cannot be applied to the operators whose heat kernel estimates fail to hold. Therefore, there is a need for a new analytic method that does not rely on the heat kernel estimates and recovers the classical results for the second order differential operators. This is the main motivation of this work.

The aim of this chapter is to establish the existence, uniqueness, pointwise upper and lower bounds, and symmetry of the Green function using purely analytic methods. Recently, an upper bound of the Green function for nonlocal Schrödinger operators with certain potentials was established by Choi and Kim [26]. However, the upper bound they obtained is not robust. Our results are new because the methods are not only purely analytic, but also robust. Furthermore, we provide an interesting example that cannot be covered by the probabilistic methods. Since this example is irrelevant to operators with kernels of variable orders, the operators under consideration in this chapter are assumed to have kernels of fixed order in order to capture the essence of the results.

The first task is to formulate a notion of Green function within the framework of weak solutions. Let $\mu = (\mu(x,\cdot))_{x\in\mathbb{R}^n}$ be a family of measures on $\mathbb{R}^n$ satisfying

$$\sup_{x\in\mathbb{R}^n} \int_{\mathbb{R}^n} \left(1 \wedge |y-x|^2\right) \mu(x,\mathrm{d}y) < +\infty. \tag{5.0.1}$$

Throughout this chapter we always assume the symmetry in the following sense:

$$\int_A \int_B \mu(x,\mathrm{d}y)\,\mathrm{d}x = \int_B \int_A \mu(x,\mathrm{d}y)\,\mathrm{d}x, \quad A, B \in \mathcal{B}(\mathbb{R}^n). \tag{5.0.2}$$

The σ-stable measure $\mu(x,\mathrm{d}y) = (2-\sigma)|y-x|^{-n-\sigma}\,\mathrm{d}y$, $\sigma \in (0,2)$ is the typical example of measure satisfying (5.0.1) and (5.0.2). Let us denote by μ_σ this measure in this chapter. As always, the constant $2-\sigma$ is used for the robust estimates.

We define for functions $u, v : \mathbb{R}^n \to \mathbb{R}$ the quantity

$$\mathcal{E}^\mu(u,v) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (u(y)-u(x))(v(y)-v(x))\mu(x,\mathrm{d}y)\,\mathrm{d}x,$$

if it is finite. For an open set $\Omega \subset \mathbb{R}^n$ and for functions $u, v : \Omega \to \mathbb{R}$ let us also define the quantity

$$\mathcal{E}^\mu_\Omega(u,v) = \int_\Omega \int_\Omega (u(y)-u(x))(v(y)-v(x))\mu(x,\mathrm{d}y)\,\mathrm{d}x, \tag{5.0.3}$$

provided that it is finite. We assume that for any compact set K and open set Ω' with $K \subset \Omega' \subset \Omega$,

$$\int_K \int_K |y-x|^2 \mu(x,\mathrm{d}y)\,\mathrm{d}x < \infty \quad \text{and} \quad \int_K \int_{\Omega\setminus\Omega'} \mu(x,\mathrm{d}y)\,\mathrm{d}x < \infty. \tag{5.0.4}$$

The assumption (5.0.4) is a sufficient and necessary condition for the quantity (5.0.3) to converge absolutely for any $u, v \in C_c^\infty(\Omega)$.

For functions $u, v : \mathbb{R}^n \to \mathbb{R}$, if $\mathcal{E}^\mu(u,v)$ is finite and $u, v = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$, then we have

$$\begin{aligned}\mathcal{E}^\mu(u,v) = &\int_\Omega \int_\Omega (u(y)-u(x))(v(y)-v(x))\mu(x,\mathrm{d}y)\,\mathrm{d}x \\ &+ \int_\Omega u(x)v(x)k_\Omega(x)\,\mathrm{d}x \\ = &\,\mathcal{E}^\mu_\Omega(u|_\Omega, v|_\Omega) + \int_\Omega u|_\Omega(x)v|_\Omega(x)k_\Omega(x)\,\mathrm{d}x,\end{aligned}$$

where $k_\Omega(x) = 2\mu(x, \mathbb{R}^n \setminus \Omega)$. Conversely, suppose that $u, v : \Omega \to \mathbb{R}$ are functions such that $\mathcal{E}^\mu_\Omega(u,v)$ finite. Define $\tilde{u} : \mathbb{R}^n \to \mathbb{R}$ by $\tilde{u}(x) = u(x)$ for $x \in \Omega$ and $\tilde{u}(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, and $\tilde{v}$ in the same way. If $\mathcal{E}^\mu(\tilde{u}, \tilde{v})$ is finite, then we have

$$\mathcal{E}^\mu(\tilde{u},\tilde{v}) = \mathcal{E}^\mu_\Omega(u,v) + \int_\Omega u(x)v(x)k_\Omega(x)\,\mathrm{d}x.$$

Therefore, for functions $u, v : \Omega \to \mathbb{R}$, by $\mathcal{E}^\mu(u,v)$ we always mean $\mathcal{E}^\mu(\tilde{u},\tilde{v})$.

Let us next define several function spaces that are appropriate for the analysis on Green functions.

Definition 5.0.1 (Function spaces)**.** Let $\Omega \subset \mathbb{R}^n$ be open and assume that a family of measures μ satisfies (5.0.1) and (5.0.2). We define the following linear spaces:

(i) $H^\mu(\mathbb{R}^n) = \{u \in L^2(\mathbb{R}^n) : \mathcal{E}^\mu(u,u) < +\infty\}$.

(ii) $H^\mu_\Omega(\mathbb{R}^n) = \{u \in H^\mu(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\}$.

We will show in Lemma 5.0.3 that the spaces $H^\mu(\mathbb{R}^n)$ and $H^\mu_\Omega(\mathbb{R}^n)$, endowed with the norm $\|u\|_{H^\mu(\mathbb{R}^n)}$, are Hilbert spaces. When $\mu = \mu_\sigma$, the space $H^{\mu_\sigma}(\mathbb{R}^n)$ is the usual fractional Sobolev space $H^{\sigma/2}(\mathbb{R}^n)$. In this regard, we denote by $H^{\sigma/2}_\Omega(\mathbb{R}^n) = W^{\sigma/2,2}_\Omega(\mathbb{R}^n)$ the space $H^{\mu_\sigma}_\Omega(\mathbb{R}^n)$, and define the following Banach spaces for $p \geq 1$:

(iii) $W^{\sigma/2,p}_\Omega(\mathbb{R}^n) = \left\{u \in W^{\sigma/2,p}(\mathbb{R}^n) : u = 0 \text{ a.e. on } \mathbb{R}^n \setminus \Omega\right\}$.

We also make use of the following linear spaces:

(iv) $L^p_{\text{weak}}(\Omega) = \{u : [u]_{L^p_{\text{weak}}(\Omega)} < +\infty\}$, where

$$[u]_{L^p_{\text{weak}}(\Omega)} = \sup_{t>0} t \left| \{x \in \Omega : |u(x)| > t\} \right|^{1/p}.$$

Remark 5.0.2. The space $H^\mu_\Omega(\mathbb{R}^n)$ in Definition 5.0.1 (ii) might not be equal to the spaces $\tilde{H}^{\sigma/2}(\Omega)$ and $H^{\sigma/2}_0(\Omega)$ in general, where $\tilde{H}^{\sigma/2}(\Omega)$ and $H^{\sigma/2}_0(\Omega)$ are defined as completions of $C^\infty_c(\Omega)$ with respect to the norms $\|\cdot\|_{H^{\sigma/2}(\mathbb{R}^n)}$ and $\|\cdot\|_{H^{\sigma/2}(\Omega)}$, respectively. In fact, two spaces $\tilde{H}^{\sigma/2}(\Omega)$ and $H^{\sigma/2}_\Omega(\mathbb{R}^n)$ coincide when Ω is a Lipschitz domain, and all spaces $\tilde{H}^{\sigma/2}(\Omega)$, $H^{\sigma/2}_\Omega(\mathbb{R}^n)$, and $H^{\sigma/2}_0(\Omega)$ coincide if $\sigma \neq 1$ is assumed additionally (cf. [62, Theorem 3.33]). However, we do not utilize both spaces $\tilde{H}^{\sigma/2}(\Omega)$ and $H^{\sigma/2}_0(\Omega)$ in this thesis.

The following lemma shows that $H^\mu(\mathbb{R}^n)$ and $H^\mu_\Omega(\mathbb{R}^n)$ are Hilbert spaces. This lemma is proved in [35, Lemma 2.3] when μ has a density.

Lemma 5.0.3. *The spaces $H^\mu(\mathbb{R}^n)$ and $H^\mu_\Omega(\mathbb{R}^n)$ are Hilbert spaces.*

Proof. Let us first show the completeness of $H^\mu(\mathbb{R}^n)$. Let (u_n) be a Cauchy sequence with respect to the norm $\|\cdot\|_{H^\mu(\mathbb{R}^n)}$. By the completeness of $L^2(\mathbb{R}^n)$, a sequence (u_n) converges to some function u in $L^2(\mathbb{R}^n)$. We may choose a subsequence (u_{n_k}) that converges to u a.e. in $\mathbb{R}^n$. By the Fatou's lemma, we have

$$\mathcal{E}^\mu(u,u) \leq \liminf_{k\to\infty} \mathcal{E}^\mu(u_{n_k}, u_{n_k}) \leq \sup_k \|u_{n_k}\|^2_{H^\mu(\mathbb{R}^n)} < +\infty,$$

which proves that $u \in H^\mu(\mathbb{R}^n)$. Again by the Fatou's lemma, we obtain

$$\mathcal{E}^\mu(u_{n_k} - u, u_{n_k} - u) \leq \liminf_{l\to\infty} \mathcal{E}^\mu(u_{n_k} - u_{n_l}, u_{n_k} - u_{n_l}) \to 0$$

as $k \to \infty$. This shows that $\|u_{n_k} - u\|_{H^\mu(\mathbb{R}^n)} \to 0$ as $k \to \infty$, and hence we have $\|u_n - u\|_{H^\mu(\mathbb{R}^n)} \to 0$ as $n \to \infty$ since (u_n) was assumed to be a Cauchy sequence. The completeness of $H^\mu_\Omega(\mathbb{R}^n)$ follows immediately. $\square$

We are now ready to define a Green function.

Definition 5.0.4. Let Ω be a bounded open set in $\mathbb{R}^n$. A function $G : \Omega \times \Omega \to [0, +\infty]$ is called a *Green function of $\mathcal{E}^\mu$ on Ω* if for each $y \in \Omega$,

$$\mathcal{E}^\mu(G(\cdot, y), \varphi) = \varphi(y) \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{5.0.5}$$

Recall that the expression (5.0.5) implies that $G(\cdot, y)$ is identified by the function $G(\cdot, y) : \mathbb{R}^n \to [0, +\infty]$ with $G(x, y) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, and that $\mathcal{E}^\mu(G(\cdot, y), \varphi)$ is finite for all $\varphi \in C_c^\infty(\Omega)$.

Let us collect several conditions on μ that are of importance for us. Throughout this chapter, we always assume that $n > \sigma$.

($\mathcal{E}_\geq$) There is a constant $c > 0$ such that

$$\mathcal{E}_B^\mu(u, u) \geq c\mathcal{E}_B^{\mu_\sigma}(u, u)$$

for every ball $B = B_r(x_0) \subset \mathbb{R}^n$ and every function $u \in L^2(B)$.

($\mathcal{E}_\leq$) There is a constant $c \geq 1$ such that

$$\mathcal{E}_B^\mu(u, u) \leq c\mathcal{E}_B^{\mu_\sigma}(u, u)$$

for every ball $B = B_r(x_0) \subset \mathbb{R}^n$ and every function $u \in L^2(B)$.

We say that μ satisfies ($\mathcal{E}$) if it satisfies ($\mathcal{E}_\geq$) and ($\mathcal{E}_\leq$).

(U1) There is a constant $c \geq 1$ such that

$$\sup_{x \in \mathbb{R}^n} \int_{\mathbb{R}^n} \left(r^2 \wedge |y - x|^2\right) \mu(x, \mathrm{d}y) \leq cr^{2-\sigma}$$

for all $r > 0$.

(U2) There exists a symmetric function K with $\mu(x, \mathrm{d}y) = K(x, y)\,\mathrm{d}y$ and

$$K(x, y) \leq c(2 - \sigma)|y - x|^{-n-\sigma}$$

for some constant $c \geq 1$.

(H) (Annulus Harnack inequality for Green function) For $M > 1$ there is a constant $c = c(M) > 1$ such that whenever $B_{Mr}(y) \subset\subset \Omega$ it holds that

$$\sup_{B_{Mr}(y)\setminus B_r(y)} G(\cdot, y) \leq c \inf_{B_{Mr}(y)\setminus B_r(y)} G(\cdot, y).$$

Remark 5.0.5. (i) The condition (U2) implies the conditions ($\mathcal{E}_{\leq}$) and (U1). Notice that in the conditions ($\mathcal{E}$) and (U1) the measures μ need not be absolutely continuous with respect to the Lebesgue measure.

(ii) The annulus Harnack inequality for Green function is implied by the usual Harnack inequality by means of a covering argument because annulus do not contain the singular point.

It is very well-known that the measure μ_σ satisfies all the conditions ($\mathcal{E}$), (U2), and (H). Let us provide other examples satisfying all or some of the above conditions.

Example 5.0.6. (i) Consider a non-degenerate σ-stable measure μ satisfying (U2) and the relative Kato condition in [11]. Then the non-degeneracy gives ($\mathcal{E}_{\geq}$). By [11, Theorem 1], any nonnegative harmonic function satisfies the Harnack inequality, which implies (H) with the help of the standard covering argument. A simple example of this measure is given by

$$\mu(x, \mathrm{d}y) = (2-\sigma)|y-x|^{-n-\sigma} \chi_{\mathcal{C}}(y-x)\, \mathrm{d}y,$$

where $\mathcal{C} = \{h \in \mathbb{R}^n : |\frac{h}{|h|} \cdot e_n| > c\}$ is a double cone.

(ii) More generally, let us consider a symmetric function $K(x, y)$ satisfying

$$K(x,y) \geq \Lambda^{-1}(2-\sigma)\left(\chi_{V^\Gamma[x]}(y) + \chi_{V^\Gamma[y]}(x)\right)|y-x|^{-n-\sigma} \quad \text{and}$$
$$K(x,y) \leq \Lambda(2-\sigma)|y-x|^{-n-\sigma},$$

where $V^\Gamma[x] = x + \Gamma(x)$ is a double cone, with apex at x and a fixed opening, that might rotate arbitrarily from point to point (see [14] for the precise definition). Then it is proved [14] that $(\mathcal{E}_{\geq})$ holds. The condition (U2) clearly holds, but it is not known whether (H) holds or not.

(iii) Another example of a non-translation invariant measure is given by

$$\mu(x, \mathrm{d}y) = (2-\sigma)\frac{a(x,y)}{|y-x|^{n+\sigma}}\,\mathrm{d}y$$

with a measurable function a which is uniformly bounded above and below away from 0. It is proved [7] that the Harnack inequality holds for nonnegative harmonic functions, from which (H) follows.

(iv) Consider a function μ satisfying the following condition: there exist $a > 1$ and $c_1, c_2 > 0$ such that every annulus $B_{a^{-k+1}} \setminus B_{a^{-k}}$, $k = 0, 1, \cdots$, contains a ball B_k with radius $c_1 a^{-k}$, such that

$$\mu(z) = c_2(2-\sigma)|z|^{-n-\sigma}, \quad z \in B_k.$$

We also assume that $\mu(z) = 0$ for $z \notin \cup_k B_k$. Then it is known [34, Proposition 6.11] that measures defined by $\mu(x, \mathrm{d}y) = \mu(y-x)\,\mathrm{d}y$ satisfy $(\mathcal{E})$. Moreover, the assumption (U2) is satisfied obviously. To our best knowledge, the condition (H) is not known.

(v) Recently, it is proved [19] that the condition $(\mathcal{E}_{\geq})$ is implied by the following mild condition: assume that there exists a function K with $\mu(x, \mathrm{d}y) = K(x,y)\,\mathrm{d}y$ and that there are constants $\delta \in (0,1)$ and $\lambda > 0$ such that for every ball $B \subset \mathbb{R}^n$ and every point $x \in B$,

$$|\{z \in B : K(x,z) \geq \lambda(2-\sigma)|z-x|^{-n-\sigma}\}| \geq \delta|B|. \tag{5.0.6}$$

It is worth noting that all the examples (i)–(iv) satisfy (5.0.6).

(vi) For $b \in (0,1)$, let $\Gamma = \big\{(x_1, x_2) \in \mathbb{R}^2 : |x_2| \geq |x_1|^b \text{ or } |x_1| \geq |x_2|^b\big\}$. Let us consider a function $\mu(z) = (2-\sigma)\mathbf{1}_{\Gamma\cap B_1}|z|^{-2-\gamma}$, where $\gamma = \sigma-1+1/b$, and measures $\mu(x, \mathrm{d}y) = \mu(y-x)\,\mathrm{d}y$. It is proved [34, Example 6] that μ satisfies $(\mathcal{E})$ and (U1). Note that (U2) is satisfied with γ, but not with σ.

(vii) Let us consider

$$\mu(x, \mathrm{d}y) = (2-\sigma)\sum_{i=1}^{n}\left(|y_i - x_i|^{-1-\sigma}\,\mathrm{d}y_i \prod_{j\neq i}\delta_{x_j}(\mathrm{d}y_j)\right).$$

It is proved [34] that μ satisfies $(\mathcal{E})$, and it is obvious that μ satisfies (U1), but not (U2). For the Harnack inequality (H) we refer to the paper [11]: they proved that the Harnack inequality holds if and only if the relative Kato condition holds. A simple computation shows that the relative Kato condition holds if and only if $\sigma > n-1$. Therefore, the condition (H) holds only when $n = 2$ and $\sigma \in (1, 2)$.

Let us present the main results in this chapter. The first result is the existence of a Green function.

Theorem 5.0.7 (Existence)**.** *Assume that μ satisfies $(\mathcal{E}_{\geq})$ for some $\sigma \in (0, 2)$. Then there exists a Green function of $\mathcal{E}^\mu$ on Ω. Moreover, any Green function G of $\mathcal{E}^\mu$ on Ω enjoys the following properties: for each $y \in \Omega$,*

$$G(\cdot, y) \in W^{\sigma/2,q}_{\Omega}(\mathbb{R}^n) \quad \text{for all } q \in [1, n/(n-\sigma/2)), \tag{5.0.7}$$

$$G(\cdot, y) \in L^{n/(n-\sigma)}_{\mathrm{weak}}(\mathbb{R}^n) \quad \text{with } [G(\cdot, y)]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\mathbb{R}^n)} \leq C, \tag{5.0.8}$$

where C depends only on n and the constant in the assumption $(\mathcal{E}_{\geq})$.

The assumption $(\mathcal{E}_{\geq})$, which is easily satisfied in various examples, is the only requirement in Theorem 5.0.7. The authors in [11] explained that for the singular measure in Example 5.0.6 (vii), the Green function is equal to $+\infty$ on the axis when $\sigma \leq (n-1)/2$. However, we would like to point out that this example is covered by Theorem 5.0.7 and hence its Green function still satisfies (5.0.7) and (5.0.8).

The next result is concerned with pointwise upper and lower bounds of Green functions, which is the essence of this chapter.

Theorem 5.0.8. *Let* $0 < \sigma_0 \leq \sigma < 2$.

(i) Assume that μ *satisfies* ($\mathcal{E}$) *and* (U2)*. Then any Green function* G *of* $\mathcal{E}^{\mu}$ *on* Ω *satisfies*

$$G(x,y) \leq C|x-y|^{\sigma-n} \quad \text{for all } x,y \in \Omega \tag{5.0.9}$$

for some constant C *depending only on* n*,* σ_0*, and the constants in the assumptions, but not on* σ*.*

(ii) Assume that μ *satisfies* ($\mathcal{E}$) *and* (U1)*, and assume that the Harnack inequality* (H) *holds. Then any Green function* G *of* $\mathcal{E}^{\mu}$ *on* Ω *satisfies*

$$G(x,y) \geq C|x-y|^{\sigma-n} \quad \text{for all } x,y \in \Omega \text{ with } |x-y| \leq \mathrm{dist}(y, \partial\Omega)/2, \tag{5.0.10}$$

for some constant C *depending only on* n*,* σ_0*, and the constants in the assumptions, but not on* σ*.*

Since the constants C in Theorem 5.0.8 stay uniform as σ approaches to 2, the pointwise bounds (5.0.9) and (5.0.10) recover the classical results [42] for second order operators as limits. As mentioned before, this is in contrast to the fact that the robustness for the heat kernel estimates is impossible.

Aside from the robustness, Theorem 5.0.8 (i) still provides a new result for the measure given in Example 5.0.6 (ii). In [73, Section 12], Schulze shows that there is a configuration Γ for which the Condition (C) in [73, Section 11.2] fails to hold. He also proved that the Condition (C) is a localized version of the assumption (UJS) in [23]. This implies, by the result of [23], that the heat kernel estimates fail for this measure. However, the Green function estimate (5.0.9) holds even for this example. This is because of the averaging effect of the Green functions.

We would like to mention that the assumption (U2) in Theorem 5.0.8 (i) can not be replaced by (U1), at least when $\sigma \leq (n-1)/2$. This can be seen from the observation in the last paragraph of [11] that the Green function for Example 5.0.6 (vii) is $+\infty$ on the axis if $\sigma \leq (n-1)/2$.

For the lower bound (5.0.10), we have Example 5.0.6 (i) and (iii) as examples. Moreover, a singular measure given in Example 5.0.6 (vii) with $\sigma > n-1$ is also covered. We do not have as much examples as upper bound and this is because we need the Harnack inequality as an assumption. We point out that, in the case of local operators, we can prove the lower bound of Green function using the weak Harnack inequality instead of the Harnack inequality, by modifying slightly the proof in [42]. A natural guess is that this will also be true for the case of nonlocal operators. However, we do not have a proof and we leave it as a future work. If this is true, Example 5.0.6 (vii) for any σ will be an example.

The last theorem we provide in this chapter is the uniqueness and the symmetry of Green function.

Theorem 5.0.9 (Uniqueness and symmetry)**.** *Assume that μ satisfies* ($\mathcal{E}$) *and* (U1)*.*

(i) There is at most one Green function of $\mathcal{E}^\mu$ on Ω satisfying (5.0.9) *and* (5.0.10)*.*

(ii) Assume the Green function G satisfies (5.0.9) *and* (5.0.10)*, then G is symmetric, i.e., $G(x,y) = G(y,x)$ for all $x, y \in \Omega$.*

Let us briefly describe the idea of proofs for main theorems. In order to establish the existence result, we essentially follow the arguments in [47], which work under very mild assumption with minor modification.

For the upper bound of Green functions we use a nonlocal version of the local boundedness results, which were established successfully in [46, 29, 30]. While the local boundedness result, together with control of some integral quantity of Green function, immediately gives the upper bound of Green functions in the case of differential operators, more investigation on the tail term should be accompanied in the case of nonlocal operators. The uniform estimates on $L^{n/(n-\sigma)}_{\mathrm{weak}}(\mathbb{R}^n)$ play an important role here.

For the lower bound of the Green function for nonlocal operators, the situation dramatically changes compared to the one for local operators. Main difficulty arises when global integral quantities are controlled by local integral quantities. We overcome the difficulty by taking appropriate test functions and splitting the double integral into a global integral quantity and a local integral quantity. The technique is mainly motivated from [29, 30], but the choice of test functions is different because test functions need to be vanished near the singularity of Green functions.

The chapter is organized as follows. In Section 5.1, we prove Theorem 5.0.7. We also provide a weak-$L^{n/(n-\sigma)}$ bound for Green functions, which will play a crucial role in the sequel. In Section 5.2 and Section 5.3 the pointwise upper and lower bounds of Green functions, respectively, are proved. Section 5.4 is devoted to the proof of Theorem 5.0.9.

5.1 Existence of a Green Function

In this section we establish the existence result. For the proof of Theorem 5.0.7, we essentially follow the arguments in [47] with minor modifications. We begin with the construction of regularized Green functions.

Lemma 5.1.1. *Assume that μ satisfies $(\mathcal{E}_{\geq})$. For each $y_0 \in \Omega$ and for any $\rho > 0$ with $B_\rho(y_0) \subset \Omega$, there exists a unique function $G_\rho(\cdot, y_0) \in H^\mu_\Omega(\mathbb{R}^n)$ satisfying*

$$\mathcal{E}^\mu(G_\rho(\cdot, y_0), \varphi) = \fint_{B_\rho(y_0)} \varphi(x)\,\mathrm{d}x \quad \text{for all } \varphi \in H^\mu_\Omega(\mathbb{R}^n). \tag{5.1.1}$$

Moreover, G_ρ is nonnegative.

Proof. $\mathcal{E}^\mu$ is a continuous bilinear form on $H^\mu_\Omega(\mathbb{R}^n)$ and $\varphi \mapsto \fint_{B_\rho(y_0)} \varphi$ is a continuous linear functional on $H^\mu_\Omega(\mathbb{R}^n)$. Moreover, the coercivity of $\mathcal{E}^\mu$ follows from [35, Lemma 2.9] by means of the assumption $(\mathcal{E}_{\geq})$. Thus, by the Lax–Milgram theorem, there exists a unique function $G_\rho(\cdot, y_0) \in H^\mu_\Omega(\mathbb{R}^n)$

satisfying (5.1.1). The nonnegativity of G_ρ can be checked in the same way as in [42]. $\square$

In order to pass the limit from (5.1.1), we next provide uniform estimates for $G_\rho(\cdot, y_0)$ in $W_\Omega^{\sigma/2,q}(\mathbb{R}^n)$ for all $q \in [1, n/(n-\sigma/2))$, which are independent of ρ. Throughout this section we write $G_\rho(x) = G_\rho(x, y_0)$ and $G(x) = G(x, y_0)$.

Proposition 5.1.2. *Assume that μ satisfies $(\mathcal{E}_\geq)$. Let $q \in [1, n/(n-\sigma/2))$, then*

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|G_\rho(y) - G_\rho(x)|^q}{|y-x|^{n+\frac{\sigma}{2}q}}\,\mathrm{d}y\mathrm{d}x \leq C$$

for some constant $C > 0$ independent of ρ.

Proof. Let $s \in (0, \sigma/n)$ and let us take $\varphi(x) = G_\rho(x)(1 + G_\rho(x)^s)^{-1/s}$ for a test function in (5.1.1). By Lemma B.0.1, to obtain

$$C(s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} sxz\left((1+G_\rho(y))^{(1-s)/2} - (1+G_\rho(x))^{(1-s)/2}\right)^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x$$
$$\leq \mathcal{E}^\mu(G_\rho, \varphi) = \fint_{B_\rho(y_0)} \varphi(x)\,\mathrm{d}x \leq 1,$$

where the last inequality follows from $\varphi \in [0,1]$. By applying the assumption $(\mathcal{E}_\geq)$ on $u = (1+G_\rho)^{(1-s)/2}$, we arrive at

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{((1+G_\rho(y))^{(1-s)/2} - (1+G_\rho(x))^{(1-s)/2})^2}{|y-x|^{n+\sigma}}\,\mathrm{d}y\,\mathrm{d}x \leq C,$$

with C independent of ρ, which is exactly the same with the estimate (19) in [47]. The remaining proof is also the same, but we shall contain the full argument here for completeness. In order to find function spaces in which G_ρ is uniformly bounded, we use the following theorem.

Theorem 5.1.3 ([69], Theorem 1 in Section 5.4.3)**.** *Assume that $\lambda > 1$ and $\lambda((n-\sigma)/2) < n$, and let $t = n/(\sigma/2 + \mu(n-\sigma)/2)$. Then*

$$\||f|^\lambda\|_{W^{\sigma/2,t}(\mathbb{R}^n)} \leq C\|f\|_{W^{\sigma/2,2}(\mathbb{R}^n)}$$

for all $f \in W^{\sigma/2,2}(\mathbb{R}^n)$.

By applying Theorem 5.1.3 to $f = (1+G_\rho)^{(1-s)/2}$ with $\lambda = 2/(1-s)$, we obtain $\|G_\rho\|_{W^{\sigma/2,t}(\mathbb{R}^n)} \leq C$ with

$$t = \frac{n}{\sigma/2 + \frac{2}{1-s}((n-\sigma)/2)} = \frac{n(1-s)}{n - \frac{\sigma}{2}(1+s)} < \frac{n}{n-\sigma/2}.$$

Note that the assumption $\lambda((n-\sigma)/2) < n$ is satisfied, provided that $s \in (0, \sigma/n)$. Taking $s = (n(1-q) + \frac{\sigma}{2}q)/(n - \frac{\sigma}{2}q)$ implies $t = q$, and hence the desired result. $\square$

Using the compactness argument, we now prove Theorem 5.0.7.

Proof of Theorem 5.0.7. Let us first prove the existence of a Green function of $\mathcal{E}^\mu$ on Ω. For any $q \in [1, n/(n-\sigma/2))$, G_ρ is uniformly bounded in $W^{\sigma/2,q}_\Omega(\mathbb{R}^n)$ by Proposition 5.1.2. Therefore, we find a sequence $\rho_k \to 0$ and a nonnegative function G such that

$$G_{\rho_k} \rightharpoonup G \quad \text{in } W^{\sigma/2,q}_\Omega(\mathbb{R}^n) \tag{5.1.2}$$

for all $q \in [1, n/(n-\sigma/2))$, and that $G_{\rho_k} \to G$ a.e. in $\mathbb{R}^n$. For each fixed $\varphi \in C_c^\infty(\Omega)$, $\mathcal{E}^\mu(\cdot, \varphi)$ is a continuous linear functional on $W^{\sigma/2,q}_\Omega(\mathbb{R}^n)$ since

$$\mathcal{E}^\mu(u,\varphi) = \langle u, L\varphi\rangle_{L^2} \leq \|u\|_{L^q(\mathbb{R}^n)} \|L\varphi\|_{L^{q'}(\mathbb{R}^n)} \leq C\|u\|_{W^{\sigma/2,q}(\mathbb{R}^n)},$$

where

$$L\varphi(x) = \text{P.V.} \int_{\mathbb{R}^n} 2(\varphi(y) - \varphi(x))\mu(x, \mathrm{d}y),$$

and $q' = q/(q-1)$ is the conjugate exponent of q. Thus, it is also continuous with respect to the weak topology and hence the equality (5.0.5) follows from the weak convergence (5.1.2). We have shown that there exists a Green function of $\mathcal{E}^\mu$ on Ω, and that it satisfies (5.0.7). In fact, we see that (5.0.7) holds for any Green function of $\mathcal{E}^\mu$ on Ω by observing that the exactly same reasoning in Proposition 5.1.2 goes through with any Green function G.

Let us next prove (5.0.8). Let G be any Green function of $\mathcal{E}^\mu$ on Ω. We take the same superlevel set and the same test function as in the proof of [47, Proposition 3.1], i.e., for $t > 0$, let $\Omega_t = \{x \in \mathbb{R}^n : G(x) > t\}$ and let $\varphi(x) = \max\{0, 1/t - 1/G(x)\}$. We claim that for all $x, y \in \mathbb{R}^n$

$$(G(y)-G(x))(\varphi(y)-\varphi(x)) \geq \left|\log\left(\frac{G(y)}{t} \vee 1\right) - \log\left(\frac{G(x)}{t} \vee 1\right)\right|^2. \tag{5.1.3}$$

Indeed, when $(x, y) \in \Omega_t \times \Omega_t$, we have $\varphi(x) = 1/t - 1/G(x)$ and $\varphi(y) = 1/t - 1/G(y)$. Thus, Lemma B.0.2 gives

$$\begin{aligned}(G(y) - G(x))(\varphi(y) - \varphi(x)) &= (G(y) - G(x))\left(\frac{1}{G(x)} - \frac{1}{G(y)}\right) \\ &\geq (\log G(y) - \log G(x))^2.\end{aligned}$$

Since $G(x) > t$ and $G(y) > t$ in this case, we arrive at (5.1.3). The case $(x, y) \in \Omega_t^c \times \Omega_t^c$ is obvious because both sides of (5.1.3) become 0. When $(x, y) \in \Omega_t^c \times \Omega_t$ we have $G(y) > t \geq G(x)$, and hence $\varphi(x) = 0$ and $\varphi(y) = 1/t - 1/G(y)$. Thus,

$$\begin{aligned}(G(y) - G(x))(\varphi(y) - \varphi(x)) &= (G(y) - G(x))\left(\frac{1}{t} - \frac{1}{G(y)}\right) \\ &\geq (G(y) - t)\left(\frac{1}{t} - \frac{1}{G(y)}\right).\end{aligned}$$

By using Lemma B.0.2 again, we obtain

$$\begin{aligned}(G(y) - t)\left(\frac{1}{t} - \frac{1}{G(y)}\right) &\geq (\log G(y) - \log t)^2 \\ &= \left(\log\left(\frac{G(y)}{t} \vee 1\right) - \log\left(\frac{G(x)}{t} \vee 1\right)\right)^2.\end{aligned}$$

A similar argument shows that (5.1.3) holds true when $(x, y) \in \Omega_t \times \Omega_t^c$. Therefore, (5.1.3) holds for all $x, y \in \mathbb{R}^n$.

We put a test function φ into (5.0.5), and then use the inequality (5.1.3).

Since $\varphi \leq 1/t$, we have

$$\begin{aligned}\frac{1}{t} &\geq \varphi(y_0) \geq \mathcal{E}(G, \varphi) \\ &\geq \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left(\log\left(\frac{G(y)}{t} \vee 1\right) - \log\left(\frac{G(x)}{t} \vee 1\right)\right)^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x.\end{aligned}$$

By the assumption $(\mathcal{E}_{\geq})$ with $u = \log(\frac{G}{t} \vee 1)$, we obtain

$$\frac{1}{t} \geq c(2-\sigma)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \left|\log\left(\frac{G(y)}{t} \vee 1\right) - \log\left(\frac{G(x)}{t} \vee 1\right)\right|^2 |x-y|^{-n-\sigma}\,\mathrm{d}y\,\mathrm{d}x.$$

Since a function $\log(G/t \vee 1)$ has a compact support in $\overline{\Omega}_t \subset \overline{\Omega}$, we can apply Proposition B.0.6 (i) to obtain

$$\begin{aligned}\frac{1}{t} &\geq C\left(\int_{\mathbb{R}^n} \left|\log\left(\frac{G(x)}{t} \vee 1\right)\right|^{2n/(n-\sigma)} \mathrm{d}x\right)^{(n-\sigma)/n} \\ &\geq C\left(\int_{\Omega_{2t}} \left|\log\left(\frac{G(x)}{t}\right)\right|^{2n/(n-\sigma)} \mathrm{d}x\right)^{(n-\sigma)/n} \geq C(\log 2)^2|\Omega_{2t}|^{(n-\sigma)/n}.\end{aligned}$$

Therefore, we conclude that

$$[G]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\mathbb{R}^n)} = \sup_{t>0} t|\Omega_t|^{(n-\sigma)/n} \leq C,$$

where C depends only on n and the constant in the assumption $(\mathcal{E}_{\geq})$. $\square$

5.2 Upper Bound of Green Functions

This section is devoted to the upper bound of Green functions. The main idea is to use a nonlocal version of the local boundedness theorem and to control the tail term by means of the estimates (5.0.8). The extension of the local boundedness theorem to the nonlocal, nonlinear framework was established successfully by Di Castro, Kuusi, and Palatucci [30], by using

Moser's iteration technique. The local behavior they obtained is of the form

$$\sup_{B_{r/2}(x_0)} u \le C \left(\fint_{B_r(x_0)} u_+^p \right)^{1/p} + \left(r^{\frac{\sigma}{2}p} \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\frac{\sigma}{2}p}} \, \mathrm{d}y \right)^{1/(p-1)} \tag{5.2.1}$$

for solutions of nonlocal operators with kernels comparable to those of fractional p-Laplacian, $p > 1$. Restricting (5.2.1) to the linear case that we are interested in implies

$$\sup_{B_{r/2}(x_0)} u \le C \left(\fint_{B_r(x_0)} u_+^2 \right)^{1/2} + r^{\sigma} \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}} \, \mathrm{d}y. \tag{5.2.2}$$

However, this is not perfectly fit to our Green functions because we need L^q-average with $q \in (1, n/(n-\sigma))$ instead of L^2-average in (5.2.2). Thus, we are going to show that (5.2.2) holds with the L^2-average replaced by the L^q-average for any $q > 1$ in a more general setting.

We first prove the following Caccioppoli-type estimates which will be used in the proof of the local boundedness.

Lemma 5.2.1 (Caccioppoli estimates). *Let $u : \mathbb{R}^n \to \mathbb{R}$ satisfy*

$$\mathcal{E}^\mu(u, \varphi) \le 0 \quad \text{for every nonnegative } \varphi \in C_c^\infty(B_\rho(x_0)). \tag{5.2.3}$$

For any $q > 1$ there exists a constant C, depending only on q, such that for any nonnegative function $\eta \in C_c^\infty(B_\rho(x_0))$

$$\begin{aligned} &\int_{B_\rho(x_0)} \int_{B_\rho(x_0)} \left(\eta(y) w^{q/2}(y) - \eta(x) w^{q/2}(x) \right)^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &\qquad \le C \int_{B_\rho(x_0)} \int_{B_\rho(x_0)} (w^q(y) + w^q(x)) |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &\qquad\quad + C \int_{B_\rho(x_0)} \int_{\mathbb{R}^n \setminus B_\rho(x_0)} w(y) \eta^2(x) w^{q-1}(x) \mu(x, \mathrm{d}y) \, \mathrm{d}x, \end{aligned}$$

where $w := (u-k)_+$ with $k \ge 0$.

Proof. In this proof, let $B = B_\rho(x_0)$. Let $\eta : \mathbb{R}^n \to [0, 1]$ be a smooth function with $\operatorname{supp} \eta \subset B$ and define $w = (u-k)_+$. We first assume that $\varphi = \eta^2 w^{q-1}$ is smooth and put φ into the equation (5.2.3). The standard regularizing argument will give the result for general case. Then we have

$$\begin{aligned} 0 \geq & \int_B \int_B (u(y) - u(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right) \mu(x, \mathrm{d}y)\, \mathrm{d}x \\ & + 2 \int_B \int_{\mathbb{R}^n \setminus B} (u(y) - u(x))(-\eta^2(x) w^{q-1}(x)) \mu(x, \mathrm{d}y)\, \mathrm{d}x =: I_1 + I_2. \end{aligned} \tag{5.2.4}$$

For the first term we observe that the integrand in I_1 satisfies

$$\begin{aligned} & (u(y) - u(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right) \\ & \geq (w(y) - w(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right). \end{aligned} \tag{5.2.5}$$

Indeed, it is obvious when $u(x), u(y) \geq k$ or $u(x), u(y) < k$. When $u(x) < k$ and $u(y) \geq k$, we have

$$\begin{aligned} & (u(y) - u(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right) \\ & = (w(y) - (u(x) - k)) \eta^2(y) w^{q-1}(y) \\ & \geq (w(y) - w(x)) \eta^2(y) w^{q-1}(y) \\ & = (w(y) - w(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right). \end{aligned}$$

We use the inequality (5.2.5) and Lemma B.0.3 to obtain

$$\begin{aligned} I_1 \geq & \int_B \int_B (w(y) - w(x)) \left(\eta^2(y) w^{q-1}(y) - \eta^2(x) w^{q-1}(x)\right) \mu(x, \mathrm{d}y)\, \mathrm{d}x \\ \geq & \frac{q-1}{32 q^2} \int_B \int_B \left(\eta(y) w^{q/2}(y) - \eta(x) w^{q/2}(x)\right)^2 \mu(x, \mathrm{d}y)\, \mathrm{d}x \\ & - 2 \left(1 \vee \frac{1}{q-1}\right) \int_B \int_B (w^q(y) + w^q(x)) |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y)\, \mathrm{d}x. \end{aligned} \tag{5.2.6}$$

For the second term we see that

$$\begin{aligned}(u(y)-u(x))(-\eta^2(x)w^{q-1}(x)) &\geq (u(y)-k)(-\eta^2(x)w^{q-1}(x))\\ &\geq -w(y)\eta^2(x)w^{q-1}(x),\end{aligned}$$

from which we estimate

$$I_2 \geq -2\int_B\int_{\mathbb{R}^n\setminus B} w(y)\eta^2(x)w^{q-1}(x)\mu(x,\mathrm{d}y)\,\mathrm{d}x. \tag{5.2.7}$$

Combining (5.2.4), (5.2.6), and (5.2.7), we obtain the desired result. □

We next prove a nonlocal version of the local boundedness theorem, which is a key ingredient for the upper bound of Green functions. The main structure of the proof is similar with the one of [30, Theorem 1.1], where the assumption (U2) is used. However, we will go into every detail to see that the pointwise upper bound assumption on kernels can be weakened in order to obtain the L^q-average instead of the L^2-average as mentioned above, and to keep track of the dependence of the constant on σ.

Theorem 5.2.2. *Let $0<\sigma_0\leq\sigma<2$ and assume that μ satisfies $(\mathcal{E}_{\geq})$ and* (U2)*. For any $q>1$, there exists a constant C, depending only on n,q,σ_0, and the constants in the assumptions, such that if u satisfies* (5.2.3)*, then*

$$\sup_{B_{r/2}(x_0)} u \leq C\left(\fint_{B_r(x_0)} u_+^q(x)\,\mathrm{d}x\right)^{1/q} + r^\sigma\int_{\mathbb{R}^n\setminus B_{r/2}(x_0)}\frac{u_+(y)}{|y-x_0|^{n+\sigma}}\,\mathrm{d}y.$$

Proof. For any $j=0,1,\cdots$, we define

$$\begin{aligned}&r_j=(1+2^{-j})\frac{r}{2},\quad \tilde{r}_j=\frac{r_j+r_{j+1}}{2},\quad B_j=B_{r_j}(x_0),\quad \tilde{B}_j=B_{\tilde{r}_j}(x_0),\\ &\eta_j\in C_c^\infty(\tilde{B}_j),\quad 0\leq\eta_j\leq 1,\quad \eta_j\equiv 1 \text{ on } B_{j+1},\quad |\nabla\eta_j|\leq 2^{j+3}/r,\\ &k_j=(1-2^{-j})K,\quad \tilde{k}_j=\frac{k_{j+1}+k_j}{2},\\ &w_j=(u-k_j)_+,\quad\text{and}\quad \tilde{w}_j=(u-\tilde{k}_j)_+,\end{aligned}$$

for some $K \geq 0$ which will be chosen later. From Lemma 5.2.1 with $\rho = r_j, \eta = \eta_j$, and $k = \tilde{k}_j$, we have

$$\begin{aligned} &\int_{B_j}\int_{B_j} \left(\eta_j(y)\tilde{w}_j^{q/2}(y) - \eta_j(x)\tilde{w}_j^{q/2}(x)\right)^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x \\ &\qquad \leq C\int_{B_j}\int_{B_j} \left(\tilde{w}_j^q(y) + \tilde{w}_j^q(x)\right) |\eta_j(y) - \eta_j(x)|^2 \mu(x, \mathrm{d}y)\,\mathrm{d}x \\ &\qquad\quad + C\int_{B_j}\int_{\mathbb{R}^n\setminus B_j} \tilde{w}_j(y)\eta_j^2(x)\tilde{w}_j^{q-1}(x)\mu(x, \mathrm{d}y)\,\mathrm{d}x =: I_1 + I_2. \end{aligned} \tag{5.2.8}$$

The symmetry (5.0.2) of μ simplifies I_1 as

$$I_1 \leq C\int_{B_j} \tilde{w}_j^q(x)\int_{B_j} |\eta_j(y) - \eta_j(x)|^2\mu(x, \mathrm{d}y)\,\mathrm{d}x.$$

Since $|\eta_j(y) - \eta_j(x)|^2 \leq 2^{2j+6}r^{-2}|y - x|^2$, the assumption (U1) yields that

$$\begin{aligned} \int_{B_j} |\eta_j(y) - \eta_j(x)|^2\mu(x, \mathrm{d}y) &\leq 2^{2j+6}r^{-2}\int_{B(x,2r_j)} |y - x|^2\mu(x, \mathrm{d}y) \\ &\leq c2^{2j+6}r^{-2}(2r_j)^{2-\sigma} \leq c2^{2j+8}r^{-\sigma}. \end{aligned}$$

We have estimated I_1 as

$$I_1 \leq C2^{2j}r^{-\sigma}\int_{B_j} \tilde{w}_j^q(x)\,\mathrm{d}x \leq C2^{2j}r^{n-\sigma}\fint_{B_j} w_j^q(x)\,\mathrm{d}x, \tag{5.2.9}$$

where we used $\tilde{w}_j \leq w_j$ in the last inequality above. Here, the constant C depends only on q, n, and the constant in the assumption (U1), but not on σ.

For I_2 we use the inequalities $\tilde{w}_j \leq w_0 = u_+$ and

$$\tilde{w}_j^{q-1} = (u - \tilde{k}_j)_+^{q-1} \leq \frac{(u - k_j)_+^q}{\tilde{k}_j - k_j} = \frac{w_j^q}{\tilde{k}_j - k_j} \leq 4K^{-1}2^j w_j^q,$$

to have

$$I_2 \leq CK^{-1}2^j\int_{\tilde{B}_j}\int_{\mathbb{R}^n\setminus B_j} u_+(y)w_j^q(x)\mu(x, \mathrm{d}y)\,\mathrm{d}x.$$

The assumption (U2) with $B_1 = B_{\tilde{r}_j}(x_0)$, $B_2 = B_{r_j}(x_0)$, $u = u_+$, and $v = w_j^q$ shows that

$$I_2 \leq C(2-\sigma)K^{-1}2^j \int_{\tilde{B}_j} \int_{\mathbb{R}^n \setminus B_j} \frac{u_+(y)}{|y-x|^{n+\sigma}} \,\mathrm{d}y \; w_j^q(x) \,\mathrm{d}x.$$

Note that here is the only place where the assumption (U2) is used in this paper. Since $\mathbb{R}^n \setminus B_j(x_0) \subset \mathbb{R}^n \setminus B_{r/2}(x_0)$ and

$$\frac{|y-x_0|}{|y-x|} \leq 1 + \frac{|x-x_0|}{|y-x|} \leq 1 + \frac{\tilde{r}_j}{r_j - \tilde{r}_j} \leq C2^j,$$

we obtain

$$\begin{aligned} I_2 &\leq CK^{-1}2^{j(n+\sigma+1)} \left(\int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}} \,\mathrm{d}y \right) \int_{B_j} w_j^q(x) \,\mathrm{d}x \\ &\leq CK^{-1}2^{j(n+\sigma+1)} r^{n-\sigma} \left(r^\sigma \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}} \,\mathrm{d}y \right) \fint_{B_j} w_j^q(x) \,\mathrm{d}x. \end{aligned} \tag{5.2.10}$$

On the other hand, applying the Proposition B.0.6 (ii) to $\eta_j \tilde{w}_j^{q/2}$ gives

$$\begin{aligned} &\left(\int_{B_{j+1}} \tilde{w}_j^{q\chi}(x) \,\mathrm{d}x \right)^{1/\chi} \\ &\leq C(2-\sigma) \left(\int_{B_j} \int_{B_j} \frac{(\eta_j(y)\tilde{w}_j^{q/2}(y) - \eta_j(x)\tilde{w}_j^{q/2}(x))^2}{|y-x|^{n+\sigma}} \,\mathrm{d}y \,\mathrm{d}x + r_j^{-\sigma} \int_{B_j} \tilde{w}_j^q \right), \end{aligned} \tag{5.2.11}$$

where $\chi = n/(n-\sigma)$. The first term in the right hand side of (5.2.11) is connected to the left hand side of (5.2.8) via the assumption ($\mathcal{E}_{\geq}$). Therefore, using $(\mathcal{E}_{\geq})$ we combine inequalities (5.2.8)–(5.2.11) as

$$\begin{aligned}&\left(\fint_{B_{j+1}} \tilde{w}_j^{q\chi}(x)\,\mathrm{d}x\right)^{1/\chi}\\&\leq C2^{2j}\fint_{B_j} w_j^q + CK^{-1}2^{j(n+\sigma+1)}\left(r^\sigma \int_{\mathbb{R}^n\setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}}\,\mathrm{d}y\right)\fint_{B_j} w_j^q.\end{aligned}$$

We use an inequality

$$\tilde{w}_j^{q\chi} = (u-\tilde{k}_j)_+^{q\chi} \geq (k_{j+1}-\tilde{k}_j)^{q(\chi-1)}(u-k_{j+1})_+^q = \left(\frac{K}{2^{j+2}}\right)^{q(\chi-1)} w_{j+1}^q$$

to obtain

$$\begin{aligned}&\left(\frac{K}{2^{j+2}}\right)^{\frac{q(\chi-1)}{\chi}} A_{j+1}^{q/\chi}\\&\leq C2^{2j}A_j^q + CK^{-1}2^{j(n+\sigma+1)}\left(r^\sigma \int_{\mathbb{R}^n\setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}}\,\mathrm{d}y\right)A_j^q,\end{aligned}$$

where we have set

$$A_j := \left(\fint_{B_j} w_j^q(x)\,\mathrm{d}x\right)^{1/q}.$$

We take

$$K \geq r^\sigma \int_{\mathbb{R}^n\setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}}\,\mathrm{d}y, \tag{5.2.12}$$

so that

$$\left(\frac{K}{2^{j+2}}\right)^{\frac{q(\chi-1)}{\chi}} A_{j+1}^{q/\chi} \leq C2^{j(n+\sigma+1)}A_j^q,$$

where $C>1$ depends only on n, q, and the constants in the assumptions. After some algebraic computations we arrive at the following recursive inequality:

$$\frac{A_{j+1}}{K} \leq \tilde{C}C_0^j\left(\frac{A_j}{K}\right)^\chi,$$

where $\tilde{C} = 2^{2(\chi-1)}C^{\chi/q}$ and $C_0 = 2^{\frac{\chi(n+\sigma+1)}{q}+\chi-1} > 1$. It will follow that $A_j \to 0$

as $j \to \infty$, provided that $A_0 \leq \tilde{C}^{-\frac{1}{\chi-1}} C_0^{-\frac{1}{(\chi-1)^2}} K$. Thus, we choose

$$K = \tilde{C}^{\frac{1}{\chi-1}} C_0^{\frac{1}{(\chi-1)^2}} A_0 + r^\sigma \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}} \,\mathrm{d}y,$$

which is in accordance with (5.2.12). Note that since $(\chi-1)^{-1} \leq 1$ and $\chi/(\chi-1) = n/\sigma \leq n/\sigma_0$, we have

$$\tilde{C}^{\frac{1}{\chi-1}} C_0^{\frac{1}{(\chi-1)^2}} = 4C^{\frac{\chi}{q(\chi-1)}} 2^{\frac{\chi}{\chi-1}\frac{n+\sigma+1}{q} + \frac{1}{\chi-1}} \leq 4C^{\frac{n}{q\sigma_0}} 2^{\frac{n(n+3)}{q\sigma_0}+1}.$$

Therefore, we deduce that

$$\sup_{B_{r/2}(x_0)} u \leq K \leq C \left(\fint_{B_r(x_0)} u_+^q(x) \,\mathrm{d}x \right)^{1/q} + r^\sigma \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{u_+(y)}{|y-x_0|^{n+\sigma}} \,\mathrm{d}y,$$

where C depends only on n, q, σ_0, and the constants in the assumptions. $\square$

We are now in a position to prove the upper bound of Green functions using Theorem 5.2.2 and the estimates (5.0.8).

Proof of Theorem 5.0.8 (i). Let G be any Green function of $\mathcal{E}^\mu$ on Ω. The assumptions ($\mathcal{E}$) and (U1) imply the assumptions (A), (B), and (D) in [34]. Thus, the interior regularity result [34, Theorem 1.3] shows that G is Hölder continuous in $\Omega \setminus N$ for any neighborhood N of the singularity.

Let $x_0, y_0 \in \Omega, x_0 \neq y_0$, and let $r := |x_0 - y_0|/2$. We first assume that $B_r(x_0) \subset \Omega$. Since $\mathcal{E}(G(\cdot, y_0), \varphi) = 0$ for all $\varphi \in C_c^\infty(B_r(x_0))$, by Theorem 5.2.2 we obtain for any $q > 1$

$$\begin{aligned} \sup_{B_{r/2}(x_0)} G(\cdot, y_0) &\leq C \left(\fint_{B_r(x_0)} G(x, y_0)^q \,\mathrm{d}x \right)^{1/q} + r^\sigma \int_{\mathbb{R}^n \setminus B_{r/2}(x_0)} \frac{G(x,y_0)}{|x-x_0|^{n+\sigma}} \,\mathrm{d}x \\ &=: I_1 + I_2, \end{aligned} \tag{5.2.13}$$

where C depends only on n, q, σ_0, and the constants in the assumptions. Since G is continuous in $B_{r/2}(x_0)$, the essential supremum in the left hand side of

(5.2.13) is realized as the supremum. Thus we have

$$G(x_0, y_0) \leq \sup_{B_{r/2}(x_0)} G(\cdot, y_0). \tag{5.2.14}$$

In particular, we choose

$$q = \frac{1}{2}\left(1 + \frac{n}{n - \sigma_0}\right) \in \left(1, \frac{n}{n - \sigma}\right), \tag{5.2.15}$$

so that the constant C in (5.2.13) depends on n, σ_0, and the constants in the assumptions only. Moreover, the choice of q as in (5.2.15) also makes the constant in the following estimate depend only on n, σ_0, and the constants in the assumptions: by the inequality (B.0.1) with $p = n/(n - \sigma)$ we have

$$\begin{aligned}
I_1 &= C|B_r|^{-1/q}\|G(\cdot, y)\|_{L^q(B_r(x_0))} \\
&\leq C\left(\frac{p}{p-q}\right)^{1/q} |B_r|^{-1/p}[G(\cdot, y_0)]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(B_r(x_0))} \\
&\leq C\left(1 + \frac{q}{p-q}\right)^{1/q} \omega_n^{-1/p} r^{\sigma-n}[G(\cdot, y_0)]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\Omega)}.
\end{aligned}$$

Since

$$p - q \geq \frac{n}{n - \sigma_0} - \frac{1}{2}\left(1 + \frac{n}{n - \sigma_0}\right) = \frac{1}{2}\frac{\sigma_0}{n - \sigma_0},$$

we obtain

$$I_1 \leq C r^{\sigma-n}[G(\cdot, y_0)]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\Omega)}.$$

This together with (5.0.8) proves

$$I_1 \leq C r^{\sigma-n}. \tag{5.2.16}$$

In order to estimate I_2, we split it into two integrals:

$$\begin{aligned} I_2 = r^\sigma & \int_{B^c_{r/2}(x_0)\cap\{G\le r^{\sigma-n}\}} \frac{G(x,y_0)}{|x-x_0|^{n+\sigma}}\,\mathrm{d}x \\ & + r^\sigma \int_{B^c_{r/2}(x_0)\cap\{G> r^{\sigma-n}\}} \frac{G(x,y_0)}{|x-x_0|^{n+\sigma}}\,\mathrm{d}x =: I_{2,1}+I_{2,2}. \end{aligned} \tag{5.2.17}$$

$I_{2,1}$ can be easily computed as follows:

$$I_{2,1} \le r^{2\sigma-n}\int_{\mathbb{R}^n\setminus B_{r/2}(x_0)} \frac{\mathrm{d}x}{|x-x_0|^{n+\sigma}} = \frac{\omega_n}{\sigma} r^{2\sigma-n}\left(\frac{r}{2}\right)^{-\sigma} \le \frac{4\omega_n}{\sigma_0} r^{\sigma-n}.$$

For $I_{2,2}$ we have

$$\begin{aligned} I_{2,2} &\le 2^{n+\sigma} r^{-n}\int_{\{G>r^{\sigma-n}\}} G(x,y_0)\,\mathrm{d}x \\ &= 2^{n+2} r^{-n}\int_{\mathbb{R}^n} \chi_{\{G>r^{\sigma-n}\}}\int_0^\infty \chi_{\{G>t\}}\,\mathrm{d}t\,\mathrm{d}x. \end{aligned}$$

By utilizing the Fubini Theorem we obtain

$$I_{2,2} \le 2^{n+2} r^{-n}\int_0^\infty \left|\left\{G(\cdot,y_0) > (r^{\sigma-n}\vee t)\right\}\right|\,\mathrm{d}t.$$

We now make use of the estimate (5.0.8) and deduce

$$\begin{aligned} I_{2,2} &\le 2^{n+2} r^{-n}\int_0^\infty \left|\left\{G(\cdot,y_0) > (r^{\sigma-n}\vee t)\right\}\right|\,\mathrm{d}t \\ &\le 2^{n+2} r^{-n}[G(\cdot,y_0)]^{n/(n-\sigma)}_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\Omega)}\int_0^\infty (r^{\sigma-n}\vee t)^{-n/(n-\sigma)}\,\mathrm{d}t \\ &\le C2^{n+2} r^{-n}\frac{n}{\sigma} r^\sigma \le C r^{\sigma-n}, \end{aligned} \tag{5.2.18}$$

where we have used $\sigma \ge \sigma_0$ in the last inequality. By combining (5.2.13)-(5.2.14) and (5.2.16)–(5.2.18), we conclude that

$$G(x_0,y_0) \le C r^{\sigma-n} \le C2^{n-\sigma_0}|x_0-y_0|^{\sigma-n}.$$

Finally, let us consider the case $B_r(x_0) \not\subset \Omega$. In this case we consider another bounded open set $\tilde{\Omega} \supset \Omega$ sufficiently large so that $B_r(x_0) \subset \tilde{\Omega}$, and let $\tilde{G}$ be a Green function of $\mathcal{E}^\mu$ on $\tilde{\Omega}$. Then

$$\mathcal{E}^\mu \left(G(\cdot, y_0) - \tilde{G}(\cdot, y_0), \varphi \right) = 0 \quad \text{for all } \varphi \in C_c^\infty(\Omega). \tag{5.2.19}$$

We know from (5.0.7) that $G(\cdot, y_0) = 0$ a.e. on $\mathbb{R}^n \setminus \Omega$ and hence

$$G(\cdot, y_0) \le \tilde{G}(\cdot, y_0) \quad \text{a.e. on } \mathbb{R}^n \setminus \Omega.$$

Thus, $\varphi := \left(G(\cdot, y_0) - \tilde{G}(\cdot, y_0) \right)_+$ is an admissible test function. Using φ in (5.2.19) we obtain

$$0 = \mathcal{E}^\mu \left(G(\cdot, y_0) - \tilde{G}(\cdot, y_0), \varphi \right) \ge \mathcal{E}^\mu(\varphi, \varphi) \ge \mathcal{E}^{\mu_\sigma}(\varphi, \varphi) \ge 0.$$

Here we used the assumption ($\mathcal{E}_\ge$). Therefore, we have $\varphi = 0$ a.e. in Ω, which in turn implies $G(\cdot, y_0) \le \tilde{G}(\cdot, y_0)$ a.e. in Ω. Since $G(\cdot, y_0) - \tilde{G}(\cdot, y_0)$ is Hölder continuous in $\Omega \setminus \{y_0\}$, we have $G(\cdot, y_0) \le \tilde{G}(\cdot, y_0)$ in $\Omega \setminus \{y_0\}$. The upper bound of G follows from the upper bound of $\tilde{G}$. □

5.3 Lower Bound of Green Functions

In this section we prove the lower bound (5.0.10). In the case of differential operators of second order we can investigate the integral in the weak formulation of Green function near the singularity by using cut-off functions. However, the situation dramatically changes for nonlocal operators because cut-off functions no longer give the integral over local regions near the singularity of Green functions. Therefore, we mainly focus on estimating these global terms by local quantities.

We begin with an estimate of a double integral of local-nonlocal nature. This quantity can be made small by assuming the local region to be very small. From now on we always denote $B_r = B_r(y_0), A_r^R = B_R \setminus B_r$, and

$G(\cdot) = G(\cdot, y_0)$ for $y_0 \in \Omega$.

Lemma 5.3.1. *Let $0 < \sigma_0 \le \sigma < 2$ and assume that μ satisfies $(\mathcal{E}_{\geq})$ and (U1). There exists a constant $\varepsilon < 1/2$, depending only on n, σ_0, and the constants in the assumptions, such that*

$$\int_{B_{\varepsilon r}(y_0)} \int_{\mathbb{R}^n \setminus B_r(y_0)} G(x, y_0) \mu(x, \mathrm{d}y) \, \mathrm{d}x \le \frac{1}{4}$$

for all $r < \operatorname{dist}(y_0, \partial\Omega)$.

Proof. If $x \in B_{\varepsilon r}$ and $y \in \mathbb{R}^n \setminus B_r$, then $|y-x| \ge |y-y_0| - |x-y_0| \ge r(1-\varepsilon)$. Thus, using the assumption (U1) we obtain

$$\begin{aligned} \int_{B_{\varepsilon r}} \int_{\mathbb{R}^n \setminus B_r} G(x) \mu(x, \mathrm{d}y) \, \mathrm{d}x &\le \int_{B_{\varepsilon r}} G(x) \int_{\mathbb{R}^n \setminus B(x, (1-\varepsilon)r)} \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &\le c \frac{\|G\|_{L^1(B_{\varepsilon r})}}{(1-\varepsilon)^\sigma r^\sigma}. \end{aligned} \tag{5.3.1}$$

We utilize the inequality (B.0.1) with $q = 1$ and $p = n/(n-\sigma)$ to get

$$\|G\|_{L^1(B_{\varepsilon r})} \le \frac{n}{\sigma} |B_{\varepsilon r}|^{\sigma/n} [G]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(B_{\varepsilon r})} \le \frac{n}{\sigma_0} \omega_n (\varepsilon r)^\sigma [G]_{L^{n/(n-\sigma)}_{\mathrm{weak}}(\Omega)}. \tag{5.3.2}$$

By combining (5.3.1), (5.3.2), and (5.0.8), we have

$$\int_{B_{\varepsilon r}} \int_{\mathbb{R}^n \setminus B_r} G(x) \mu(x, \mathrm{d}y) \, \mathrm{d}x \le C \left(\frac{\varepsilon}{1-\varepsilon} \right)^\sigma,$$

where C depends only on n, σ_0, and the constants in the assumptions. Note that here we needed the assumption $(\mathcal{E}_{\geq})$ to make use of the estimate (5.0.8). Thus, we take

$$\varepsilon < \frac{1}{2} \min \left\{ 1, (4C)^{-1/\sigma_0} \right\},$$

so that $C(\varepsilon/(1-\varepsilon))^\sigma \le C(2\varepsilon)^{\sigma_0} < 1/4$. □

The next lemma shows how the integral over a global region can be controlled by a local quantity. The method used in the following lemma is motivated by [29, Lemma 4.2]. The difference is that we use a cut-off function whose support is in an annulus near the singularity of Green functions. More precisely, we use a cut-off function $\eta : \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\eta \in [0,1], \quad \eta = 1 \text{ in } A^r_{\varepsilon r}, \quad \eta = 0 \text{ in } \mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}, \quad \text{and} \quad |\nabla \eta| \le \frac{4}{\varepsilon} r^{-1}, \tag{5.3.3}$$

where ε is the constant from Lemma 5.3.1.

Lemma 5.3.2. *Let $0 < \sigma_0 \le \sigma < 2$ and assume that μ satisfies $(\mathcal{E}_{\ge})$ and* (U1)*. Let ε be the constant in Lemma 5.3.1, and let η be the cut-off function satisfying* (5.3.3)*. There exists a constant C, depending only on n, σ_0, and the constants in the assumptions, such that*

$$\int_{\mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}(y_0)} \int_{A^{3r/2}_{\varepsilon r/2}(y_0)} G(x, y_0) \eta^2(y) \mu(x, \mathrm{d}y) \, \mathrm{d}x \le C r^{n-\sigma} \sup_{A^{3r/2}_{\varepsilon r/2}(y_0)} G(\cdot, y_0). \tag{5.3.4}$$

for all $r < \mathrm{dist}(y_0, \partial\Omega)/2$. In particular,

$$\int_{\mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}(y_0)} \int_{A^r_{\varepsilon r}(y_0)} G(x, y_0) \mu(x, \mathrm{d}y) \, \mathrm{d}x \le C r^{n-\sigma} \sup_{A^{3r/2}_{\varepsilon r/2}(y_0)} G(\cdot, y_0). \tag{5.3.5}$$

Before we give a proof of Lemma 5.3.2 let us prove a useful estimates of a cut-off function η satisfying (5.3.3).

Lemma 5.3.3. *Under the same setting as in Lemma 5.3.2,*

$$\int_{A^{3r/2}_{\varepsilon r/2}(y_0)} \int_{\mathbb{R}^n} |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \le C r^{n-\sigma},$$

for some C depending only on n, σ_0, and the constants in the assumptions.

Proof. It follows from (U1) that

$$\int_{\mathbb{R}^n} |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \le \int_{\mathbb{R}^n} \left(1 \wedge \frac{16}{\varepsilon^2 r^2} |y - x|^2 \right) \mu(x, \mathrm{d}y) \le C r^{-\sigma},$$

where C depends on ε and the constant c in the assumption (U1) only. Since

ε depends only on n, σ_0, and the constants in the assumptions, we have

$$\int_{A^{3r/2}_{\varepsilon r/2}(y_0)} \int_{\mathbb{R}^n} |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \leq Cr^{n-\sigma}.$$

Note that the use of the assumption $(\mathcal{E}_{\geq})$ is hidden in the fact that ε depends on n, σ_0, and the constants in the assumptions only. □

We are ready to prove Lemma 5.3.2 using Lemma 5.3.3.

Proof of Lemma 5.3.2. Since we know that G is continuous in $\Omega \setminus \{y_0\}$, G is locally bounded in $\Omega \setminus \{y_0\}$. We set

$$k = \sup_{A^{3r/2}_{\varepsilon r/2}(y_0)} G(\cdot, y_0) < +\infty.$$

Putting $\varphi = (G - 2k)\eta^2$ into (5.0.5) and using the symmetry (5.0.2) of a measure, we have

$$\begin{aligned} 0 = &\int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} (G(y) - G(x))(\varphi(y) - \varphi(x)) \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &+ 2 \int_{\mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} (G(y) - G(x)) \varphi(y) \mu(x, \mathrm{d}y) \, \mathrm{d}x =: I_1 + I_2. \end{aligned} \tag{5.3.6}$$

Write $w = G - 2k$. For $x, y \in A^{3r/2}_{\varepsilon r/2}$ with $\eta(y) \geq \eta(x)$, we have

$$\begin{aligned} &(G(y) - G(x))(\varphi(y) - \varphi(x)) \\ &= (w(y) - w(x))^2 \eta^2(y) + (w(y) - w(x)) w(x) (\eta^2(y) - \eta^2(x)) \\ &\geq (w(y) - w(x))^2 \eta^2(y) - 2|w(y) - w(x)||w(x)|\eta(y)|\eta(y) - \eta(x)| \\ &\geq -|w(x)|^2 |\eta(y) - \eta(x)|^2 \geq -4k^2 |\eta(y) - \eta(x)|^2, \end{aligned}$$

and the same inequality holds for $x, y \in A^{3r/2}_{\varepsilon r/2}$ with $\eta(x) \leq \eta(y)$. This inequality and Lemma 5.3.3 yields

$$I_1 \geq -4k^2 \int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \geq -Ck^2 r^{n-\sigma}, \tag{5.3.7}$$

where C depends only on n, σ_0, and the constants in the assumptions.

For I_2, we split the integral into two parts as

$$\begin{aligned}I_2 = 2&\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}(G(y)-G(x))(G(y)-2k)\chi_{\{G(x)\geq k\}}\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x\\ &+2\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}(G(y)-G(x))(G(y)-2k)\chi_{\{G(x)<k\}}\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x\\ =&I_{2,1}+I_{2,2}.\end{aligned}$$

Since $G(y)\leq k$ in $A^{3r/2}_{\varepsilon r/2}$, we have

$$\begin{aligned}(G(y)-G(x))(G(y)-2k)\chi_{\{G(x)\geq k\}} &= (G(x)-G(y))(2k-G(y))\chi_{\{G(x)\geq k\}}\\ &\geq (G(x)-k)k\end{aligned}\tag{5.3.8}$$

and

$$\begin{aligned}(G(y)-G(x))(G(y)-2k)\chi_{\{G(x)<k\}} &= -(G(y)-G(x))(2k-G(y))\chi_{\{G(x)<k\}}\\ &\geq -2k(G(y)-G(x))_+\chi_{\{G(x)<k\}}\geq -2k^2.\end{aligned}\tag{5.3.9}$$

Using (5.3.8), we obtain

$$\begin{aligned}I_{2,1}\geq 2k&\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}G(x)\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x\\ -2k^2&\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x,\end{aligned}\tag{5.3.10}$$

and using (5.3.9), we obtain

$$I_{2,2}\geq -4k^2\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x.\tag{5.3.11}$$

We combine the estimates (5.3.10) and (5.3.11), and then use Lemma 5.3.3

to get

$$I_2 \geq 2k \int_{\mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} G(x)\eta^2(y)\mu(x, \mathrm{d}y)\, \mathrm{d}x - Ck^2 r^{n-\sigma}. \tag{5.3.12}$$

The inequality (5.3.4) is obtained by combining (5.3.6), (5.3.7), and (5.3.12). The second assertion (5.3.5) immediately follows from (5.3.3). □

The next lemma corresponds to the estimate of L^2-norm of the gradient of Green functions in the case of second order differential operators. Global terms arising from the weak formulation of Green function now can be controlled by using Lemma 5.3.2.

Lemma 5.3.4. *Let $0 < \sigma_0 \leq \sigma < 2$ and assume that μ satisfies $(\mathcal{E}_{\geq})$ and* (U1). *Let ε be the constant in Lemma 5.3.1. There exists a constant C, depending only on n, σ_0, and the constants in the assumptions, such that*

$$\int_{A^{3r/2}_{\varepsilon r/2}(y_0)} \int_{A^{r}_{\varepsilon r}(y_0)} (G(y, y_0) - G(x, y_0))^2 \mu(x, \mathrm{d}y)\, \mathrm{d}x \leq C r^{n-\sigma} \sup_{A^{3r/2}_{\varepsilon r/2}(y_0)} G^2(\cdot, y_0).$$

for all $r <$ $\mathrm{dist}(y_0, \partial\Omega)/2$.

Proof. As before, we know that $\sup_{A^{3r/2}_{\varepsilon r/2}} G < \infty$. Let $\eta : \mathbb{R}^n \to \mathbb{R}$ be a cut-off function satisfying (5.3.3). We put $\varphi = G\eta^2$ into (5.0.5) and then use the symmetry assumption (5.0.2) to have

$$\begin{aligned} 0 = &\int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} (G(y) - G(x))(\varphi(y) - \varphi(x))\mu(x, \mathrm{d}y)\, \mathrm{d}x \\ &+ 2 \int_{\mathbb{R}^n \setminus A^{3r/2}_{\varepsilon r/2}} \int_{A^{3r/2}_{\varepsilon r/2}} (G(y) - G(x))\varphi(y)\mu(x, \mathrm{d}y)\, \mathrm{d}x =: I_1 + I_2. \end{aligned} \tag{5.3.13}$$

We utilize Lemma B.0.4 and Lemma 5.3.3 to estimate I_1 as

$$\begin{aligned}
I_1 &\geq \frac{1}{4}\int_{A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}(G(y)-G(x))^2(\eta^2(y)+\eta^2(x))\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad -4\int_{A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}(G^2(y)+G^2(x))|\eta(y)-\eta(x)|^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq \frac{1}{4}\int_{A^{3r/2}_{\varepsilon r/2}}\int_{A^{r}_{\varepsilon r}}(G(y)-G(x))^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\quad -8\Big(\sup_{A^{3r/2}_{\varepsilon r/2}} G^2\Big)\int_{A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}}|\eta(y)-\eta(x)|^2\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq \frac{1}{4}\int_{A^{3r/2}_{\varepsilon r/2}}\int_{A^{r}_{\varepsilon r}}(G(y)-G(x))^2\mu(x,\mathrm{d}y)\,\mathrm{d}x - Cr^{n-\sigma}\sup_{A^{3r/2}_{\varepsilon r/2}} G^2.
\end{aligned} \tag{5.3.14}$$

For I_2 note that we have

$$\begin{aligned}
I_2 &\geq -2\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}} G(x)G(y)\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&\geq -2\Big(\sup_{A^{3r/2}_{\varepsilon r/2}} G\Big)\int_{\mathbb{R}^n\setminus A^{3r/2}_{\varepsilon r/2}}\int_{A^{3r/2}_{\varepsilon r/2}} G(x)\eta^2(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x.
\end{aligned}$$

Therefore, (5.3.4) gives

$$I_2 \geq -Cr^{n-\sigma}\sup_{A^{3r/2}_{\varepsilon r/2}} G^2. \tag{5.3.15}$$

The proof is finished by putting (5.3.14) and (5.3.15) into (5.3.13). □

We now provide the proof of the lower bound of Green functions by gathering pieces of integrals in the preceding lemmas.

Proof of Theorem 5.0.8 (ii). Let $x_0, y_0 \in \Omega, x_0 \neq y_0, r = |x_0 - y_0| \leq d(y_0)/2$, and let ε be the constant in Lemma 5.3.1. Let $\varphi : \mathbb{R}^n \to \mathbb{R}$ be a cut-off function satisfying

$$\varphi \in [0,1], \quad \varphi = 1 \text{ in } B_{\varepsilon r}(y_0), \quad \varphi = 0 \text{ outside } B_r(y_0), \quad \text{and} \quad |\nabla\varphi| \leq 4r^{-1}.$$

The equality (5.0.5) with φ gives

$$\begin{aligned}
1 = &\, 2\int_{B_{\varepsilon r}}\int_{\mathbb{R}^n\setminus B_r}(G(y)-G(x))(-1)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+ 2\int_{B_{\varepsilon r/2}}\int_{A^r_{\varepsilon r}}(G(y)-G(x))(\varphi(y)-1)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+ 2\int_{\mathbb{R}^n\setminus B_{3r/2}}\int_{A^r_{\varepsilon r}}(G(y)-G(x))\varphi(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
&+ \iint_{\left(A^{3r/2}_{\varepsilon r/2}\times A^r_{\varepsilon r}\right)\cup\left(A^r_{\varepsilon r}\times A^{3r/2}_{\varepsilon r/2}\right)}(G(y)-G(x))(\varphi(y)-\varphi(x))\mu(x,\mathrm{d}y)\,\mathrm{d}x \\
= &\, I_1+I_2+I_3+I_4.
\end{aligned} \tag{5.3.16}$$

We use Lemma 5.3.1 to have

$$I_1 \le 2\int_{B_{\varepsilon r}}\int_{\mathbb{R}^n\setminus B_r} G(x)\mu(x,\mathrm{d}y)\,\mathrm{d}x \le \frac{1}{2}. \tag{5.3.17}$$

For the second term we observe that

$$(G(y)-G(x))(\varphi(y)-1) = G(x)(1-\varphi(y)) - G(y)(1-\varphi(y)) \le G(x).$$

Thus, by means of (5.3.5) we obtain

$$I_2 \le 2\int_{B_{\varepsilon r/2}}\int_{A^r_{\varepsilon r}} G(x)\mu(x,\mathrm{d}y)\,\mathrm{d}x \le Cr^{n-\sigma}\sup_{A^{3r/2}_{\varepsilon r/2}} G. \tag{5.3.18}$$

We next estimate the third term. Note that we have

$$I_3 \le 2\int_{\mathbb{R}^n\setminus B_{3r/2}}\int_{A^r_{\varepsilon r}} G(y)\mu(x,\mathrm{d}y)\,\mathrm{d}x \le 2\left(\sup_{A^r_{\varepsilon r}} G\right)\int_{A^r_{\varepsilon r}}\int_{\mathbb{R}^n\setminus B_{3r/2}}\mu(x,\mathrm{d}y)\,\mathrm{d}x,$$

where we used the symmetry (5.0.2) in the last inequality above. By utilizing the assumption (U1) we obtain

$$\int_{A^r_{\varepsilon r}}\int_{\mathbb{R}^n\setminus B_{3r/2}}\mu(x,\mathrm{d}y)\,\mathrm{d}x \le \int_{A^r_{\varepsilon r}}\int_{\mathbb{R}^n\setminus B(x,r/2)}\mu(x,\mathrm{d}y)\,\mathrm{d}x \le Cr^{n-\sigma},$$

which yields

$$I_3 \le Cr^{n-\sigma} \sup_{A^r_{\varepsilon r}} G. \tag{5.3.19}$$

For the last term we make use of the symmetry (5.0.2) and the Hölder inequality to have

$$\begin{aligned} I_4 &\le 2 \int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^r_{\varepsilon r}} |G(y) - G(x)||\eta(y) - \eta(x)| \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &\le 2 \left(\int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^r_{\varepsilon r}} |G(y) - G(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \right)^{1/2} \\ &\quad \times \left(\int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^r_{\varepsilon r}} |\eta(y) - \eta(x)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \right)^{1/2}. \end{aligned}$$

It follows from Lemma 5.3.4, together with an estimate

$$\begin{aligned} &\int_{A^{3r/2}_{\varepsilon r/2}} \int_{A^r_{\varepsilon r}} |\eta(x) - \eta(y)|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \\ &\le 16 r^{-2} \int_{A^{3r/2}_{\varepsilon r/2}} \int_{B(x, 5r/2)} |y - x|^2 \mu(x, \mathrm{d}y) \, \mathrm{d}x \le Cr^{n-\sigma}, \end{aligned}$$

that

$$I_4 \le Cr^{n-\sigma} \sup_{A^{3r/2}_{\varepsilon r/2}} G. \tag{5.3.20}$$

Combining all estimates (5.3.16)–(5.3.20) we deduce that

$$1 \le Cr^{n-\sigma} \sup_{A^{3r/2}_{\varepsilon r/2}} G(\cdot, y_0) + \frac{1}{2},$$

or equivalently,

$$\sup_{A^{3r/2}_{\varepsilon r/2}} G(\cdot, y_0) \ge C|x_0 - y_0|^{n-\sigma}.$$

By the Harnack inequality (H), we have

$$\inf_{A^{3r/2}_{\varepsilon r/2}} G(\cdot, y_0) \ge C|x_0 - y_0|^{\sigma-n}.$$

Note that the above essential infimum is realized as the infimum since G is continuous in $A^{3r/2}_{\varepsilon r/2}$. Since $x_0 \in A^{3r/2}_{\varepsilon r/2}(y_0)$, we get the desired result. $\square$

5.4 Uniqueness and Symmetry of the Green function

This section is devoted to the uniqueness of Green function satisfying the pointwise upper and lower bounds.

Proof of Theorem 5.0.9. Let $G, \tilde{G}$ be Green functions of $\mathcal{E}^\mu$ on Ω, and let $y \in \Omega$ be fixed. We write $G(x) = G(x, y)$ and $\tilde{G}(x) = \tilde{G}(x, y)$. Since G and $\tilde{G}$ enjoy (5.0.9) and (5.0.10), we find a constant $c > 0$ such that

$$v(x) := G(x) - c\tilde{G}(x) \geq 0 \quad \text{for all } x \in B_{\rho_0}(y),$$

where $\rho_0 = d(y)/2$. A function v_- is an admissible test function since v vanishes outside Ω. Thus, we have

$$0 = v_-(y) = \mathcal{E}^\mu(v, v_-) = \mathcal{E}^\mu(v_+, v_-) - \mathcal{E}^\mu(v_-, v_-) \leq -\mathcal{E}^{\mu_\sigma}(v_-, v_-) \leq 0,$$

where we used the assumption $(\mathcal{E}_\geq)$. This implies $v_- \equiv 0$ a.e. in $\mathbb{R}^n$, or equivalently, $v \geq 0$ a.e. in $\mathbb{R}^n$. Recall that the interior regularity result [34, Theorem 1.3] shows that v is Hölder continuous in $\Omega \setminus B_{\rho_0}(y)$, which implies $v \geq 0$ in Ω. Thus, we set

$$c_0 := \sup\{c : G - c\tilde{G} \geq 0 \text{ in } \Omega\},$$

and claim that $c_0 = 1$. Note that we may assume that $c_0 \leq 1$. Indeed, if $c_0 > 1$, then $\tilde{c}_0 := \sup\{c : \tilde{G} - cG \geq 0 \text{ in } \Omega\} \leq 1$ since otherwise we have $G \geq c_0\tilde{G} \geq c_0\tilde{c}_0 G > G$ in Ω, a contradiction. In this case, we may consider $\tilde{c}_0$ instead of c_0 and then prove that $\tilde{c}_0 = 1$. Once we have proved $\tilde{c}_0 = 1$, the possibility $c_0 > 1$ is actually excluded because in this case we have the same contradiction $\tilde{G} \geq G \geq c_0\tilde{G} > \tilde{G}$ in Ω.

We will show that $c_0 = 1$ by excluding the possibility $c_0 < 1$. If $c_0 < 1$, then the function $u := (G - c_0\tilde{G})/(1 - c_0)$ satisfies

$$\mathcal{E}^\mu(u, \varphi) = \varphi(y) \quad \text{for all } \varphi.$$

Thus, u is a Green function of $\mathcal{E}^\mu$ on Ω. By the assumption it enjoys (5.0.10), that is, we have $u \geq C|x - y|^{\sigma - n}$ for $|x - y| \leq \rho_0$. Using the upper bound of $\tilde{G}$, we obtain $G \geq (c_0 + \delta)\tilde{G}$ in $B_{\rho_0}(y)$ for some small constant $\delta > 0$. The same argument as above shows that $G \geq (c_0 + \delta)\tilde{G}$ in Ω, which contradicts the maximality of c_0. Therefore, c_0 must be 1. $\square$

The symmetry of the Green function follows from the symmetry (5.0.2) of measure μ. The proof relies on the regularity estimates in [34].

Proof of Theorem 5.0.9 (ii). Let $x, y \in \Omega$, $x \neq y$. We recall that the unique Green function G of $\mathcal{E}^\mu$ on Ω is constructed by a sequence of regularized Green functions. That is, we have sequence $\rho_i < |x-y|/3$ and corresponding regularized Green functions $G_{\rho_i}(\cdot, y)$ that converges a.e. to $G(\cdot, y)$. Let $\tau_j < |x - y|/3$ be another sequence corresponding to regularized Green functions $G_{\tau_j}(\cdot, x)$ that converges a.e. to $G(\cdot, x)$. Then we have

$$\fint_{B_{\rho_i}(y)} G_{\tau_j}(\cdot, x) = \mathcal{E}(G_{\rho_i}(\cdot, y), G_{\tau_j}(\cdot, x)) = \fint_{B_{\tau_j}(x)} G_{\rho_i}(\cdot, y) =: a_{ij}.$$

Since $G_{\tau_j}(\cdot, x)$ weakly converges to $G(\cdot, x)$ in $W^{\sigma/2,q}_\Omega(\mathbb{R}^n)$ for all $q \in [1, n/(n-\sigma/2))$, by letting $\tau_j \to 0$ we have

$$\lim_{j\to\infty} a_{ij} = \fint_{B_{\rho_i}(y)} G(\cdot, x).$$

Since $G(\cdot, y)$ is continuous on $B_{\tau_j}(x)$, we obtain

$$\lim_{i\to\infty} \lim_{j\to\infty} a_{ij} = G(y, x).$$

In the same way we have

$$\lim_{j\to\infty}\lim_{i\to\infty} a_{ij} = G(x,y).$$

Now, [34, Theorem 1.3] says that we can bound the Hölder norm of $G_{\rho_i}(\cdot, y)$ on $B_{\tau_j}(x)$ independent of i, which means that the double sequence a_{ij} converges uniformly in j with respect to i. It completes the proof. $\square$

Chapter 6

Boundary Regularity

We turn our attention to regularity properties of solutions up to boundary in this chapter. Let us consider the Dirichlet problem

$$\begin{cases} Lu = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D, \end{cases} \tag{6.0.1}$$

where D is a bounded $C^{1,1}$ open set in $\mathbb{R}^n$ and $L = L_\varphi$ is an operator given by (1.2.2), which coincides with an infinitesimal generator (1.1.1) of an isotropic unimodal Lévy process when u is a bounded C^2 function. We will show, under the assumption that φ satisfies the weak scaling condition (2.1.1) at zero, that there exists a unique solution u of (6.0.1) and that u is controlled by $V(d_D(x))$ near the boundary ∂D, where V is the renewal function that behaves like the square root of φ and $d_D(x) = \mathrm{dist}(x, \mathbb{R}^n \setminus D)$. We then further investigate a Hölder regularity of the quotient $u/V(d_D)$ up to the boundary. The results in this chapter are based on the joint work in [48].

Regularity estimates of u/d^s for the fractional Laplacian-type operators have been studied in [41, 40, 63, 64, 65, 66, 67, 68]. It was first proved in [63] that if D is of $C^{1,1}$ and $f \in L^\infty(D)$, then $u/d^s \in C^{s-\varepsilon}(\overline{D})$. They also established in [64] that if D is of $C^{2,\gamma}$ and $f \in C^\gamma(\overline{D})$, then $u/d^s \in C^{\gamma+s}(\overline{D})$ in more general contexts of fully nonlinear equations. Higher regularity of u/d^s, provided that D and f are regular enough, was provided in [40, 41] for elliptic pseudo-differential operators satisfying the s-transmission property.

The aim of this chapter is to extend the result of [63] to operators with kernels having variable orders. In [63], as a generalization of a barrier function $d_D(x)$ in the case of local operators, Ros-Oton and Serra used $d_D^s(x)$ as a barrier for the fractional Laplacian, which still has a simple form because of a nice scaling property of the fractional Laplacian. On the other hand, the kernels under consideration in this chapter only have a weak scaling condition which allows nontrivial boundary behaviors different from d_D^s. To overcome the lack of a simple barrier, we will consider the renewal function V of the ladder height process defined at (6.1.1). Moreover, we track down u in every scale to find scale invariant uniform estimates only with the weak scaling condition at zero.

Let us recall that the characteristic exponent Φ of an isotropic unimodal Lévy process X is given by

$$\Phi(z) = \int_{\mathbb{R}^n} \left(e^{iz\cdot x} - 1 - iz \cdot x\mathbf{1}_{\{|x|\le 1\}}\right) J(|x|)\,\mathrm{d}x$$

with the Lévy measure $J(|x|)\,\mathrm{d}x$, and that the infinitesimal generator of X is given by

$$Au(x) = \frac{1}{2}\int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))\,\frac{J(1)}{|y|^n\varphi(|y|)}\,\mathrm{d}y,$$

with $\varphi(r) = \frac{J(1)}{J(r)}r^{-n}$, for bounded C^2 functions u. Throughout the chapter we assume that the characteristic exponent Φ satisfies the weak scaling condition at infinity with constants $0 < \sigma_1 \le \sigma_2 < 2$ and $a_1 \ge 1$, i.e.,

$$a_1^{-1}\left(\frac{R}{r}\right)^{\sigma_1} \le \frac{\Phi(R)}{\Phi(r)} \le a_1\left(\frac{R}{r}\right)^{\sigma_2} \quad \text{for all } 1 \le r \le R, \tag{6.0.2}$$

where we regarded Φ as a function defined on $\mathbb{R}_+$ since it is isotropic. We also assume that the density J satisfies

$$J(r) \le a_2 J(r+1) \text{ for all } r > 0 \quad \text{and} \quad r \mapsto -\frac{J'(r)}{r} \text{ is non-increasing.} \tag{6.0.3}$$

It is known [9] that $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \le 1$ with a comparison constant depending on n. Thus, the assumption (6.0.2) reads as

$$a^{-1}\left(\frac{R}{r}\right)^{\sigma_1} \le \frac{\varphi(R)}{\varphi(r)} \le a\left(\frac{R}{r}\right)^{\sigma_2} \quad \text{for all } 0 < r \le R \le 1, \tag{6.0.4}$$

for some constant $a = a(a_1, n) \ge 1$.

We say that $D \subset \mathbb{R}^n$ (when $n \ge 2$) is a $C^{1,1}$ open set if there exist a localization radius $R_0 > 0$ and a constant $\Lambda > 0$ such that for every $z \in \partial D$ there exist a $C^{1,1}$ function $\Psi = \Psi_z : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\Psi(0) = 0$, $\nabla\Psi(0) = 0$, $\|\nabla\Psi\|_\infty \le \Lambda$, and $|\nabla\Psi(x) - \nabla\Psi(w)| \le \Lambda|x - w|$, and an orthonormal coordinate system CS_z of $z = (z_1, \cdots, z_{n-1}, z_n) := (z', z_n)$ with origin at z such that $D \cap B(z, R_0) = \{y = (y', y_n) \in B(0, R_0) \text{ in } CS_z : y_n > \Psi(y')\}$. The pair (R_0, Λ) is called the $C^{1,1}$ characteristics of the open set D. Note that a $C^{1,1}$ open set D with characteristics (R_0, Λ) may be unbounded and disconnected, and the distance between two distinct components of D is at least R_0. By a $C^{1,1}$ open set in $\mathbb{R}$ with a characteristic $R_0 > 0$, we mean an open set that can be written as the union of disjoint intervals so that the infimum of the lengths of all these intervals is at least R_0 and the infimum of the distances between these intervals is at least R_0.

The first result in this chapter is the solvability of the Dirichlet problem (6.0.1) and generalized Hölder estimates up to the boundary of the solution when D is a bounded $C^{1,1}$ open set in $\mathbb{R}^n$.

Theorem 6.0.1 (Hölder estimates up to the boundary)**.** *Assume that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^n$ and that X is an isotropic pure jump Lévy process satisfying* (6.0.2) *and* (6.0.3)*. If $f \in C(D)$, then there exists a unique viscosity solution u of* (6.0.1)*. Moreover, $u \in C^{\bar\phi}(\mathbb{R}^n)$ and*

$$\|u\|_{C^{\bar\phi}(\mathbb{R}^n)} \le C\|f\|_{L^\infty(D)}, \tag{6.0.5}$$

where $\bar\phi(r) := \varphi(r)^{1/2}$, for some constant $C > 0$ depending only on n, D, and Φ.

For the solvability of the Dirichlet problem (6.0.1), we use the potential operator which is the inverse of the infinitesimal generator A of X. Since we are concerned with viscosity solutions for L, we need to investigate a relation between viscosity solutions for L and solutions for A. For the generalized Hölder estimates, we make use of the estimates on the transition density and its spatial derivatives.

It is well-known that $\bar{\phi}$ is comparable to the renewal function V, which will be defined in Section 6.1. This implies that the solution u of (6.0.1) is $C^V(\mathbb{R}^n)$ by Theorem 6.0.1. Moreover, we will see, in Remark 6.3.6, that C^V regularity of u up to the boundary is optimal. Hence, it is of importance to study the regularity of $u/V(d_D)$ up to the boundary. This is provided in the next result.

Theorem 6.0.2 (Boundary estimates)**.** *Assume that D is a bounded $C^{1,1}$ open set in $\mathbb{R}^n$ and that X is an isotropic pure jump Lévy process satisfying* (6.0.2) *and* (6.0.3)*. If $f \in C(D)$ and u is the viscosity solution of* (6.0.1)*, then $u/V(d_D)|_D$ can be continuously extended to $\overline{D}$ and $u/V(d_D) \in C^\alpha(\overline{D})$ for some $\alpha \in (0,1)$. Moreover, we have*

$$\left\| \frac{u}{V(d_D)} \right\|_{C^\alpha(\overline{D})} \leq C\|f\|_{L^\infty(D)}$$

for some constant $C > 0$. The constants α and C depend only on n, D, and Φ.

For the boundary estimates we follow the boundary Harnack method in [63], which was first developed in [55] for the second order differential equations. In other words, we are going to control the oscillation of the function $u/V(d_D)$ near the boundary using the standard barrier argument. The novelty of this work is in the construction of barriers. The difficulty mainly comes from the facts that the operator L is not scale invariant and that V is not regular enough.

The chapter is organized as follows. In Section 6.1, we define the renewal function V which will describe the behavior of solutions near the boundary.

Some useful properties of the renewal function that are inherited from the weak scaling condition (6.0.2) are collected. We provide the solvability of the Dirichlet problem (6.0.1) in Section 6.2 by using the potential operator and investigating a relation between viscosity solutions for L and solutions for A. Section 6.3 is devoted to the generalized Hölder regularity estimates for the solution, finishing the proof of Theorem 6.0.1. We next construct a barrier function in Section 6.4 using the renewal function in order to prove Theorem 6.0.2. The proof of Theorem 6.0.2 will be given in Section 6.5.

6.1 The Renewal Function

Let us define a renewal function that will play a fundamental role in describing the behavior of solutions to (6.0.1) near the boundary.

Let $Z = (Z_t)_{t\geq 0}$ be a one-dimensional Lévy process with a characteristic exponent $\Phi(|z|)$ and $M_t := \sup\{Z_s : 0 \leq s \leq t\}$ be the supremum of Z. Let $L = (L_t)_{t\geq 0}$ be a local time of $M_t - Z_t$ at 0, which satisfies

$$L_t = \int_0^t \mathbf{1}_{\{M_t = Z_t\}}(s)\, \mathrm{d}s.$$

Since $t \mapsto L_t$ is non-decreasing and continuous with probability 1, we can define the right-continuous inverse of L by

$$L^{-1}(t) := \inf\{s > 0 : L(s) > r\}.$$

The mapping $t \mapsto L^{-1}(t)$ is non-decreasing and right-continuous a.s. The process $L^{-1} = (L_t^{-1})_{t\geq 0}$ with $L_t^{-1} = L^{-1}(t)$ is called the *ascending ladder time process* of Z. The *ascending ladder height process* $H = (H_t)_{t\geq 0}$ is defined as

$$H_t := \begin{cases} M_{L_t^{-1}} (= Z_{L_t^{-1}}) & \text{if } L_t^{-1} < \infty, \\ \infty & \text{otherwise.} \end{cases}$$

See [36] for the details. We define the *renewal function* of the ladder height process H with respect to Φ by

$$V(x) = \int_0^\infty \mathbb{P}(H_s \le x)\,\mathrm{d}s, \quad x \in \mathbb{R}. \tag{6.1.1}$$

It is known that $V(x) = 0$ if $x \le 0$, $V(\infty) = \infty$, and V is strictly increasing and differentiable on $(0, \infty)$. Thus, there exists the inverse function V^{-1} : $[0, \infty) \to [0, \infty)$.

The most important property of the renewal function is that the function defined by $w(x) := V(x_n)$ solves

$$\begin{cases} Lw = 0 & \text{in} \quad \mathbb{R}^n_+, \\ w = 0 & \text{in} \quad \mathbb{R}^n \setminus \mathbb{R}^n_+. \end{cases} \tag{6.1.2}$$

We refer the reader to [43, Theorem 3.3] for the proof. Using this fact, we will construct barriers in Section 6.4 for the proof of Theorem 6.0.2.

In the following lemmas, we collect some properties of the renewal function. The first lemma shows that the assumption (6.0.2) induces the weak scaling condition on the renewal function.

Lemma 6.1.1. *We have*

$$V(r)^2 \asymp \varphi(r) \quad \text{in } 0 < r \le 1. \tag{6.1.3}$$

Moreover, there exist constants $a_3, a_4 \ge 1$ such that

$$a_3^{-1} \left(\frac{R}{r}\right)^{\sigma_1/2} \le \frac{V(R)}{V(r)} \le a_3 \left(\frac{R}{r}\right)^{\sigma_2/2}, \quad 0 < r \le R \le 1, \tag{6.1.4}$$

and

$$a_4^{-1} \left(\frac{T}{t}\right)^{2/\sigma_2} \le \frac{V^{-1}(T)}{V^{-1}(t)} \le a_4 \left(\frac{T}{t}\right)^{2/\sigma_1}, \quad 0 < t \le T \le V(1). \tag{6.1.5}$$

Proof. By [9, Corollary 3] and [10, Proposition 2.4], we have

$$(V(r))^{-2} \asymp \Phi(r^{-1}), \quad r > 0, \tag{6.1.6}$$

with a comparison constant depending only on n. Combining (6.1.6) with $\Phi(r^{-1})^{-1} \asymp \varphi(r)$ in $0 < r \leq 1$, we conclude (6.1.3). Moreover, by (6.1.3) and (6.0.4), we have (6.1.4). Using [9, Remark 4], we also obtain the weak scaling condition (6.1.5) of the inverse function V^{-1}. □

As in Lemma 2.1.1, we prove the following inequalities using the weak scaling property (6.1.4) for V. Since we are not concerned with the robustness, we do not keep track of the dependence of C on σ_1 and σ_2. The inequality (6.1.8) is given in [10, Lemma 3.5] but we include the proof for the completeness.

Lemma 6.1.2. *There is a constant $C > 0$ such that*

$$\int_0^r \frac{1}{V(s)}\,\mathrm{d}s \leq C\frac{r}{V(r)}, \quad \int_0^r \frac{V(s)}{s}\,\mathrm{d}s \leq CV(r), \tag{6.1.7}$$

and

$$\int_r^\infty \frac{V(s)}{s\varphi(s)}\,\mathrm{d}s \leq C\frac{1}{V(r)}, \tag{6.1.8}$$

for every $0 < r \leq 1$.

Proof. By the weak scaling property (6.1.4), we have

$$\int_0^r \frac{1}{V(s)}\,\mathrm{d}s = \int_0^r a_3 \left(\frac{r}{s}\right)^{\sigma_2/2} \frac{1}{V(r)}\,\mathrm{d}s \leq C\frac{r}{V(r)}$$

and

$$\int_0^r \frac{V(s)}{s}\,\mathrm{d}s = \int_0^r a_3 \left(\frac{s}{r}\right)^{\sigma_1/2} \frac{V(r)}{s}\,\mathrm{d}s \leq CV(r),$$

which prove (6.1.7). For (6.1.8), let

$$\mathcal{P}(r) := \int_{\mathbb{R}} \left(1 \wedge \frac{|x|^2}{r^2}\right) J(x)\,\mathrm{d}x$$

be the Pruitt function of X. By [9, (6) and Lemma 1] and (6.1.3), we have

$$\mathcal{P}(r) \le C\varphi(r)^{-1} \le CV(r)^{-2}, \quad r > 0.$$

Let us recall that the scale function $\overline{C}_\varphi(r)$ is given by (2.1.2). We observe that

$$\overline{C}_\varphi(r) = C\int_{B(0,r)^c} \left(1 \wedge \frac{|x|^2}{r^2}\right) J(|x|)\,\mathrm{d}x \le C\mathcal{P}(r) \le CV(r)^{-2}. \tag{6.1.9}$$

Thus, using the integration by parts and (6.1.9), we have

$$\begin{aligned}\int_r^\infty \frac{V(s)}{s\varphi(s)}\,\mathrm{d}s &= \int_r^\infty V(s)\,\mathrm{d}(-\overline{C}_\varphi)(s) \\ &= V(r)\overline{C}_\varphi(r) - \lim_{s\to\infty} V(s)\overline{C}_\varphi(s) + \int_r^\infty V'(s)\overline{C}_\varphi(s)\,\mathrm{d}s \le \frac{C}{V(r)},\end{aligned}$$

which concludes (6.1.8). □

The next lemma is concerned with the estimates for derivatives of V, whose proof is given in [43, Proposition 3.1] and [58, Theorem 1.2].

Lemma 6.1.3. *Assume that X is an isotropic pure jump Lévy process satisfying (6.0.2) and (6.0.3). Then V is twice-differentiable on $(0,\infty)$ and satisfies*

$$|V''(r)| \le C\frac{V'(r)}{r\wedge 1} \quad \text{and} \quad V'(r) \le C\frac{V(r)}{r\wedge 1}, \; r > 0, \tag{6.1.10}$$

for some constant $C > 0$.

6.2 Potential Operator and Dirichlet Problem

This section is devoted to the solvability of the Dirichlet problem (6.0.1). Since we are going to use the potential operator which is the inverse of the generator $-A$ of X, we need to investigate a relation between viscosity solutions of (6.0.1) and solutions of the problem

$$\begin{cases} Au = f & \text{in } D, \\ u = 0 & \text{in } \mathbb{R}^n \setminus D. \end{cases} \tag{6.2.1}$$

In [3], Baeumer, Luke, and Meerschaert discussed the domains and value of the operators L and A. We apply the strategies in [3] to our setting and obtain some related properties. We then establish the existence and uniqueness results for the Dirichlet problems (6.0.1) and (6.2.1) with the help of the comparison principle in Section 2.3. Moreover, we will see that these two solutions coincide under some conditions.

Throughout this chapter, we assume that $D \subset \mathbb{R}^n$ is a bounded $C^{1,1}$ open set with $\mathrm{diam}(D) \leq 1$. Let us define the potential operator from now on. The isotropic unimodal Lévy process X possesses a transition density p satisfying

$$P_t f(x) = \mathbb{E}^x[f(X_t)] = \int_{\mathbb{R}^n} f(y) p(t, |x-y|) \, \mathrm{d}y.$$

Let $\tau_D := \inf\{t > 0 : X_t \notin D\}$ be the first exit time of D by X. We define a subprocess $X^D = (X_t^D)_{t \geq 0}$, which is called a *killed process of X upon D*, by $X_t^D = X_t$ when $t < \tau_D$ and $X_t^D = \partial$ when $t \geq \tau_D$, where ∂ is a cemetery point. Since X has the transition density p, X^D also possesses the transition density p_D which is given by

$$p_D(t, x, y) = p(t, |x-y|) - \mathbb{E}^x[p(t - \tau_D, |X_{\tau_D} - y|); \tau_D < t].$$

The transition density p_D is also called the *Dirichlet heat kernel*. Its transition semigroup $(P_t^D)_{t \geq 0}$ is represented by

$$P_t^D f(x) := \mathbb{E}^x[f(X_t^D)] = \int_D f(y) p_D(t, x, y) \, \mathrm{d}y.$$

We define a *Green function of* X^D by

$$G^D(x,y) = \int_0^\infty p_D(t,x,y)\,\mathrm{d}t,$$

for $x, y \in D$ with $x \neq y$, which is known [22, Theorem 1.5] to be finite for $x \neq y$. We define a *potential operator* R^D for X^D as

$$R^D f(x) := \int_0^\infty \int_D p_D(t,x,y)f(y)\,\mathrm{d}y\,\mathrm{d}t.$$

Using definitions of P_t^D and G^D, it can be written by

$$R^D f(x) = \int_{D\setminus\{x\}} G^D(x,y)f(y)\,\mathrm{d}y = \int_0^\infty P_t^D f(x)\,\mathrm{d}t. \tag{6.2.2}$$

We will see that R^D acts as the inverse of $-A$.

Let us now investigate a relation between viscosity solutions to (6.0.1) and solutions to (6.2.1). Let $\mathcal{D} := \mathcal{D}(D) := \{u \in C_0(D) : Au \in C(D)\}$ be the domain of operator A, where $C_0(D)$ denotes the set of continuous functions on D that tend to zero as $x \in D$ approaches the boundary. Recall that by [3, Lemma 2.6] we have

$$Au(x) = Lu(x) \tag{6.2.3}$$

for any $u \in C^2(x) \cap C_0(\mathbb{R}^n)$, $x \in D$. We first show that the function $u = -R^D f$ satisfies (6.2.1) when f is continuous.

Lemma 6.2.1. *Let $f \in C(D)$ and define $u = -R^D f$. Then, u is a solution of* (6.2.1)*.*

Proof. Let us first claim that for any $v \in C_0(D)$ and $x \in D$,

$$Av(x) = \lim_{t\searrow 0} \frac{P_t^D v(x) - v(x)}{t}. \tag{6.2.4}$$

To prove (6.2.4), we basically follow the lines of the proof of [3, Theorem 2.3]. Note that our domain $\mathcal{D}$ is slightly different from the one in [3]. Indeed, for $v \in C_0(D)$ and $x \in D$, we have

$$P_t^D v(x) - P_t v(x) = \mathbb{E}^x v(X_t^D) - \mathbb{E}^x v(X_t) = -\mathbb{E}^x[v(X_t)\mathbf{1}_{\{\tau_D < t\}}],$$

and hence,

$$\begin{aligned}\frac{P_t^D v(x) - v(x)}{t} - \frac{P_t v(x) - v(x)}{t} &= -\frac{\mathbb{E}^x[v(X_t)\mathbf{1}_{\{\tau_D < t\}}]}{t} \\ &= \frac{\mathbb{E}^x[(v(X_{\tau_D}) - v(X_t))\,\mathbf{1}_{\{\tau_D < t\}}]}{t}.\end{aligned}$$

Meanwhile, by the strong Markov property we obtain

$$\left|\mathbb{E}^x\left[(v(X_{\tau_D}) - v(X_t))\,\mathbf{1}_{\{\tau_D < t\}}\right]\right| \le \mathbb{E}^x\left[\left|\mathbb{E}^{X_{\tau_D}}[v(X_0) - v(X_{t-\tau_D})]\right|\mathbf{1}_{\{\tau_D < t\}}\right].$$

Since $v \in C_0(D)$ is uniformly continuous, by the stochastic continuity of Lévy process, for any $\varepsilon > 0$ there is a $\delta = \delta(\varepsilon) > 0$ such that

$$|\mathbb{E}^z[v(X_s)] - v(z)| < \varepsilon$$

for any $z \in D$ and $0 < s \le \delta$. Thus, we have

$$\left|\mathbb{E}^x[(v(X_{\tau_D}) - v(X_t))\,\mathbf{1}_{\{\tau_D < t\}}]\right| \le \varepsilon \mathbb{P}^x(\tau_D < t)$$

for $0 < t \le \delta$. Since D is open, for any $x \in D$ we find a radius $r_x > 0$ such that $B(x, r_x) \subset D$. Using [12, Theroem 5.1 and Proposition 2.27 (d)] there exists $M > 0$ such that

$$\frac{\mathbb{P}^x(\tau_D < t)}{t} \le \frac{\mathbb{P}^x(\tau_{B(x,r_x)} < t)}{t} \le M \quad \text{for all } t > 0.$$

Combining above inequalities, we obtain that

$$\begin{aligned}\lim_{t\searrow 0}\left|\frac{P_t^D v(x)-v(x)}{t}-Av(x)\right| &= \lim_{t\searrow 0}\left|\frac{P_t^D v(x)-v(x)}{t}-\frac{P_t v(x)-v(x)}{t}\right| \\ &\le \varepsilon \lim_{t\searrow 0}\frac{\mathbb{P}^x[\tau_D<t]}{t}\le \varepsilon M.\end{aligned}$$

Since $\varepsilon>0$ is arbitrarily, this concludes the claim.

Let us now finish the lemma. Note that $u=0$ in $\mathbb{R}^n\setminus D$ by the definition of R^D. Therefore, by using the claim and the semigroup property $P_s^D P_t^D = P_{s+t}^D$, we have for $x\in D$,

$$\begin{aligned}Au(x)=A(-R^D f)(x) &= -\lim_{t\searrow 0}\frac{P_t^D(R^D f)(x)-R^D f(x)}{s} \\ &= -\lim_{t\searrow 0}\frac{1}{t}\left[P_t^D\left(\int_0^\infty P_s^D f(\cdot)\,\mathrm{d}s\right)(x)-\int_0^\infty P_s^D f(x)\,\mathrm{d}s\right] \\ &= \lim_{t\searrow 0}\frac{1}{t}\left(-\int_0^\infty P_{t+s}^D f(x)\,\mathrm{d}s+\int_0^\infty P_s^D f(x)\,\mathrm{d}s\right) \\ &= \lim_{t\searrow 0}\frac{1}{t}\left(-\int_t^\infty P_s^D f(x)\,\mathrm{d}s+\int_0^\infty P_s^D f(x)\,\mathrm{d}s\right) \\ &= \lim_{t\searrow 0}\frac{1}{t}\int_0^t P_s^D f(x)\,\mathrm{d}s = f(x),\end{aligned}$$

which finishes the proof. □

The relation between solutions for A and viscosity solutions for L is exhibited in the following lemma.

Lemma 6.2.2. *Assume that $f\in C(D)$ and $u\in\mathcal{D}$ satisfy $Au=f$ in D. Then, u is a viscosity solution of $Lu=f$.*

Proof. Let $x\in D$ and let $v\in C^2(\mathbb{R}^n)$ be a test function satisfying $v(x)=u(x)$ and $v(y)>u(y)$ for all $y\in\mathbb{R}^n\setminus\{x\}$. Since $P_t^D v(x)\ge P_t^D u(x)$ for every $t>0$, we have

$$Av(x)=\lim_{t\searrow 0}\frac{P_t^D v(x)-v(x)}{t}\ge \lim_{t\searrow 0}\frac{P_t^D u(x)-u(x)}{t}=Au(x).$$

Thus, it follows from $Lv(x) = Av(x)$ that $Lv(x) \geq f(x)$, which shows that u is a viscosity subsolution of $Lu = f$. The same argument proves that u is a viscosity supersolution of $Lu = f$, concluding the lemma. $\square$

The existence and uniqueness parts of Theorem 6.0.1 are immediate from Lemma 6.2.1, Lemma 6.2.2, and Theorem 2.3.6.

Theorem 6.2.3. *Let $f \in C(D)$. Then $u = -R^D f \in \mathcal{D}$ is the unique solution of* (6.2.1). *Moreover, u is the unique viscosity solution of* (6.0.1).

6.3 Hölder Regularity up to Boundary

In this section, we prove the rest part of Theorem 6.0.1—the generalized Hölder estimates up to boundary—by using the Dirichlet heat kernel estimates from [22, Corollary 1.6] and [59, Thoerem 1.1 and 1.2]. We reformulate them here for the purpose of this section.

Theorem 6.3.1. *Let $D \subset \mathbb{R}^n$ be a bounded $C^{1,1}$ open set with* $\mathrm{diam}(D) \leq 1$. *Let X be an isotropic unimodal Lévy process satisfying* (6.0.2) *and* (6.0.3), *and let $p_D(t,x,y)$ be the Dirichlet heat kernel for X on D. Then p_D is differentiable with respect to x for each $y \in D$ and $t > 0$, and satisfies the following estimates:*

(a) For any $(t,x,y) \in (0,1] \times D \times D$,

$$p_D(t,x,y) \leq C \left(1 \wedge \frac{V(d_D(x))}{t^{1/2}}\right) \left(1 \wedge \frac{V(d_D(y))}{t^{1/2}}\right) p\left(t, |x-y|/4\right)$$

and

$$|\nabla_x p_D(t,x,y)| \leq C \left(\frac{1}{d_D(x) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{t})}\right) p_D(t,x,y).$$

(b) For any $(t,x,y) \in [1,\infty) \times D \times D$,

$$p_D(t,x,y) \leq C e^{-\lambda_1 t} V(d_D(x)) V(d_D(y))$$

and

$$|\nabla_x p_D(t,x,y)| \le C\left(\frac{1}{d_D(x)\wedge 1} \vee \frac{1}{V^{-1}(1)}\right) p_D(t,x,y),$$

where $-\lambda_1 < 0$ *is the largest eigenvalue of the generator of* $X^{B(0,1)}$.

The constant C *depends on* n, D, *and* Φ.

The reformulation in Theorem 6.3.1 may need some explanations. From the original estimates in [22, Corollary 1.6] and [59, Themreo 1.1 and 1.2], we have used $d_D(x) \vee d_D(y) \le \mathrm{diam}(D) \le 1$, $V(r) \asymp \varphi(r)^{1/2}$ in $0 < r \le 1$, and $\frac{1}{V^{-1}(\sqrt{t})} \asymp \varphi^{-1}(t)$ in Theorem 6.3.1. In addition, the estimate in [22, Corollary 1.6] is of the form

$$p_D(t,x,y) \le ce^{-\lambda(D)t} V(d_D(x))V(d_D(y)),$$

where $-\lambda(D) < 0$ is the largest eigenvalue of the generator of X^D. Using [37, (6.4.14) and Lemma 6.4.5], we have

$$\lambda(D) = \inf\left\{ -\int_{\mathbb{R}^n} Lu(x)u(x)\,\mathrm{d}x : \|u\|_{L^2} = 1,\, \mathrm{supp}\, u \subset D \right\}.$$

Thus we may obtain $\lambda_1 \le \lambda(D)$ and it implies the estimates in Theorem 6.3.1 (b).

Since (6.0.5) holds trivially for an unbounded function f, we assume that $f \in L^\infty(D)$ in the rest of this section. We point out that the continuity of f is not required in the propositions and remarks in the rest of this section. Let us prove interior Hölder estimates for $R^D f$.

Proposition 6.3.2. *Let* $f \in L^\infty(D)$ *and let* $B_r(x_0) \subset D$ *be such that* $d_D(x_0) \le 2r$. *Then* $R^D f \in C^V(\overline{B_{r/2}(x_0)})$ *and*

$$\|R^D f\|_{C^V(\overline{B_{r/2}(x_0)})} \le C\left(\|f\|_{L^\infty(D)} + \|R^D f\|_{C(\overline{B_r(x_0)})} \right) \tag{6.3.1}$$

for some constant $C > 0$ *depending on* n, D, *and* Φ.

Proof. Let $x, y \in B_{r/2}(x_0)$ and let $h = |y - x|$. Using (6.2.2), we have

$$\frac{|R^D f(y) - R^D f(x)|}{V(h)} \le \int_0^\infty \frac{|P_s^D f(y) - P_s^D f(x)|}{V(h)} \, \mathrm{d}s. \tag{6.3.2}$$

We split the integral in the right-hand side of (6.3.2) into three integrals over intervals $[0, V(h)V(r))$, $[V(h)V(r), V(r)^2)$, and $[V(r)^2, \infty)$, and denote by I_1, I_2, and I_3 these integrals, respectively.

Let us first estimate I_1. It follows easily from $|P_s^D f(x)| \le \|f\|_{L^\infty(D)}$ that

$$I_1 \le \int_0^{V(h)V(r)} \frac{2\|f\|_{L^\infty(D)}}{V(h)} \, \mathrm{d}s \le 2V(r)\|f\|_{L^\infty(D)}. \tag{6.3.3}$$

To estimate I_2, we will use Theorem 6.3.1 (a). For $s \le V(r)^2$ and for any $x^* \in B_{r/2}(x_0)$, by using (6.1.5), we have

$$\frac{1}{d_D(x^*) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \le \frac{C}{V^{-1}(\sqrt{s})}.$$

Thus, by Theorem 6.3.1 (a), we obtain

$$\begin{aligned} |\nabla_x P_s^D f(x^*)| &\le C \left(\frac{1}{d_D(x^*) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \right) \|P_s^D f\|_{L^\infty(D)} \\ &\le \frac{C}{V^{-1}(\sqrt{s})} \|f\|_{L^\infty(D)}, \end{aligned}$$

and hence,

$$\begin{aligned} I_2 &= \int_{V(h)V(r)}^{V(r)^2} \frac{|P_s^D f(y) - P_s^D f(x)|}{V(h)} \, \mathrm{d}s \\ &\le \frac{h}{V(h)} \int_{V(h)V(r)}^{V(r)^2} |\nabla_x P_s^D f(x^*)| \, \mathrm{d}s \\ &\le C \frac{h}{V(h)} \|f\|_{L^\infty(D)} \int_{V(h)V(r)}^{V(r)^2} \frac{1}{V^{-1}(\sqrt{s})} \, \mathrm{d}s, \end{aligned} \tag{6.3.4}$$

where x^* is some point on the line segment between x and y. The integral in the right hand-side of (6.3.4) can be further estimated by using Lemma 6.1.3

as

$$\int_{V(h)V(r)}^{V(r)^2} \frac{\mathrm{d}s}{V^{-1}(\sqrt{s})} = 2\int_{\varepsilon}^{r} \frac{V(t)V'(t)}{t}\,\mathrm{d}t \le C\int_{\varepsilon}^{r} \left(\frac{V(t)}{t}\right)^2 \mathrm{d}t, \qquad (6.3.5)$$

where $\varepsilon := V^{-1}(V(h)^{1/2}V(r)^{1/2})$. We use the weak scaling property (6.1.4) of V to obtain

$$\frac{V(t)}{V(\varepsilon)} \le a_3\left(\frac{t}{\varepsilon}\right)^{\sigma_2/2} \le a_3\frac{t}{\varepsilon}, \quad t \ge \varepsilon. \qquad (6.3.6)$$

Thus, by (6.3.6) and (6.1.7), we have

$$\int_{\varepsilon}^{r} \left(\frac{V(t)}{t}\right)^2 \mathrm{d}t \le CV(r)\frac{V(\varepsilon)}{\varepsilon}. \qquad (6.3.7)$$

Combining (6.3.4), (6.3.5), and (6.3.7), and then using the weak scaling property (6.1.4) again, we conclude that

$$\begin{aligned} I_2 &\le CV(r)\frac{h}{V(h)}\frac{V(\varepsilon)}{\varepsilon}\|f\|_{L^\infty(D)} \\ &\le CV(r)\left(\frac{h}{\varepsilon}\right)^{1-\sigma_2/2}\|f\|_{L^\infty(D)} \le CV(r)\|f\|_{L^\infty(D)}, \end{aligned} \qquad (6.3.8)$$

where we have used that $\varepsilon \ge h$.

For I_3, we divide the integral into two integrals with $V(r)^2 \le s \le 1$ and $s > 1$. If $V(r)^2 \le s \le 1$, then for any $x^* \in B_{r/2}(x_0)$,

$$\frac{1}{d_D(x^*) \wedge 1} \vee \frac{1}{V^{-1}(\sqrt{s})} \vee \frac{1}{V^{-1}(1)} \le \frac{1}{r}.$$

Thus, by Theorem 6.3.1 (a), the assumption $d_D(x_0) \le 2r$, and the weak scaling property (6.1.4), we have

$$\begin{aligned} |\nabla_x p_D(s, x^*, y)| &\le \frac{C}{r}\left(1 \wedge \frac{V(d_D(x^*))}{s^{1/2}}\right)\left(1 \wedge \frac{V(d_D(y))}{s^{1/2}}\right)p(s, |x^* - y|/4) \\ &\le \frac{C}{r}\frac{V(r)}{s^{1/2}}p(s, |x^* - y|/4). \end{aligned}$$

Therefore, we obtain

$$
\begin{aligned}
|P_s^D f(y) - P_s^D f(x)| &= h|\nabla_x P_s^D f(x^*)| \\
&\le h\|f\|_{L^\infty(D)} \int_D |\nabla_x p_D(s, x^*, y)| \,\mathrm{d}y \\
&\le Ch\frac{V(r)}{rs^{1/2}}\|f\|_{L^\infty(D)} \int_D p\left(s, \frac{|x^* - y|}{4}\right) \mathrm{d}y \\
&\le Ch\frac{V(r)}{rs^{1/2}}\|f\|_{L^\infty(D)},
\end{aligned} \tag{6.3.9}
$$

where x^* is some point on the line segment between x and y. In the last inequality of (6.3.9), we used the fact that $\int_{\mathbb{R}^n} p(s, y/4) \,\mathrm{d}y = 4^n$.

If $s \ge 1$, then by Theorem 6.3.1 (b), we have for any $x^* \in B_{r/2}(x_0)$,

$$
|\nabla_x p_D(s, x^*, y)| \le \frac{C}{r} e^{-\lambda_1 s} V(d_D(x^*)) V(d_D(y)) \le \frac{CV(r)}{r} e^{-\lambda_1 s}, \tag{6.3.10}
$$

where we used $d_D(x_0) \le 2r$, $d_D(y) \le \mathrm{diam}(D) \le 1$, and the weak scaling property (6.1.4). The estimate (6.3.10) yields that

$$
\begin{aligned}
|P_s^D f(y) - P_s^D f(x)| &\le h\|f\|_{L^\infty(D)} \int_D |\nabla_x p_D(s, x^*, y)| \,\mathrm{d}y \\
&\le Ch\frac{V(r)}{r}\|f\|_{L^\infty(D)} e^{-\lambda_1 s},
\end{aligned} \tag{6.3.11}
$$

where x^* is some point on the line segment between x and y. We now combine (6.3.9) and (6.3.11) to estimate

$$
\begin{aligned}
I_3 &= \int_{V(r)^2}^1 \frac{|P_s^D f(y) - P_s^D f(x)|}{V(h)} \,\mathrm{d}s + \int_1^\infty \frac{|P_s^D f(y) - P_s^D f(x)|}{V(h)} \,\mathrm{d}s \\
&\le C\frac{V(r)}{r}\frac{h}{V(h)} \left(\int_{V(r)^2}^1 \frac{1}{\sqrt{s}} \,\mathrm{d}s + \int_1^\infty e^{-\lambda_1 s} \,\mathrm{d}s \right) \|f\|_{L^\infty(D)} \\
&\le C(2 - 2V(r) + \lambda_1^{-1} e^{-\lambda_1}) \|f\|_{L^\infty(D)} \le C\|f\|_{L^\infty(D)}.
\end{aligned} \tag{6.3.12}
$$

We finally combine (6.3.3), (6.3.8), and (6.3.12) to obtain

$$
[R^D f]_{C^V(\overline{B_{r/2}(x_0)})} \le C(1 + V(r))\|f\|_{L^\infty(D)} \le C(1 + V(1))\|f\|_{L^\infty(D)}. \tag{6.3.13}
$$

The desired result follows from (6.3.13) and the definition of C^V-norm. $\square$

We next capture a behavior of the function $R^D f$ near the boundary by using various estimates in [10, 22, 43]. Especially, the second assertion in the following lemma will provide the optimality of Theorem 6.0.1.

Lemma 6.3.3. *There is a constant $C = C(n, D, \Phi) > 0$ such that*

$$|R^D f(x)| \leq C\|f\|_{L^\infty(D)} V(\mathrm{diam}(D)) V(d_D(x))$$

for any $f \in L^\infty(D)$ and $x \in D$. Moreover, if we further assume that $f > 0$ in D, then for any $\delta > 0$ there exists a constant $c = c(n, D, \Phi, \delta) > 0$ such that

$$R^D f(x) \geq cV(d_D(x)) \int_{D_\delta} f(y)\, \mathrm{d}y \tag{6.3.14}$$

for every $x \in D$, where $D_\delta := \{y \in D : d_D(y) > \delta\}$.

Proof. By [10, Theorem 4.6] and (6.1.3),

$$\int_D G^D(x, y)\, \mathrm{d}y \leq CV(\mathrm{diam}(D))V(d_D(x)).$$

Thus, we obtain

$$|R^D f(x)| \leq \|f\|_{L^\infty(D)} \int_D G^D(x, y)\, \mathrm{d}y \leq C\|f\|_{L^\infty(D)} V(\mathrm{diam}(D))V(d_D(x)),$$

which proves the first assertion.

For (6.3.14), we assume that $f > 0$ and that D_δ is nonempty. Let us first consider a point $x \in D$ with $d_D(x) < \delta/2$. Then, for any $y \in D_\delta$, we have $\delta/2 \leq |x - y| \leq \mathrm{diam}(D)$ and $\delta \leq d_D(y)$. Thus, by using [43, Theorem 1.6], the proof of [22, Theorem 7.3 (i)], (6.0.4), and (6.1.4), we have

$$\begin{aligned} G^D(x,y) &\geq c\frac{\varphi(|x-y|)}{|x-y|^n}\left(1\wedge\frac{\varphi(d_D(x))}{\varphi(|x-y|)}\right)^{1/2}\left(1\wedge\frac{\varphi(d_D(y))}{\varphi(|x-y|)}\right)^{1/2} \\ &\geq c\frac{\varphi(|x-y|)}{|x-y|^n}\left(\frac{\varphi(d_D(x))}{\varphi(|x-y|)}\right)^{1/2}\left(1\wedge\frac{\varphi(\delta)}{\varphi(\operatorname{diam}(D))}\right)^{1/2} \\ &\geq cV(d_D(x))\frac{V(|x-y|)}{|x-y|^n} \\ &\geq cV(d_D(x))\frac{V(\delta/2)}{\operatorname{diam}(D)^n} \geq cV(d_D(x)). \end{aligned}$$

Therefore, we obtain

$$R^D f(x) \geq \int_{D_\delta} G^D(x,y)f(y)\,\mathrm{d}y \geq cV(d_D(x))\int_{D_\delta} f(y)\,\mathrm{d}y. \tag{6.3.15}$$

Let us next consider a point x with $d_D(x) \geq \delta/2$. Then for $y \in D_\delta$ with $|x-y| \geq \delta/8$, we have

$$\begin{aligned} G^D(x,y) &\geq c\frac{\varphi(|x-y|)}{|x-y|^n}\left(1\wedge\frac{\varphi(d_D(x))}{\varphi(|x-y|)}\right)^{1/2}\left(1\wedge\frac{\varphi(d_D(y))}{\varphi(|x-y|)}\right)^{1/2} \\ &\geq c\frac{\varphi(\delta/8)}{\operatorname{diam}(D)^n}\left(1\wedge\frac{\varphi(\delta/2)}{\varphi(\operatorname{diam}(D))}\right)^{1/2}\left(1\wedge\frac{\varphi(\delta)}{\varphi(\operatorname{diam}(D))}\right)^{1/2} \geq c. \end{aligned}$$

On the other hand, if $y \in D_\delta$ with $|x-y| < \delta/8$, then by using [22, Proposition 3.3] we have

$$G^D(x,y) \geq c\int_{\varphi(|x-y|)}^{\varphi(d_D(x)\wedge d_D(y))} (\varphi^{-1}(t))^{-n}\,\mathrm{d}t \geq c\int_{\varphi(\delta/8)}^{\varphi(\delta/2)} (\varphi^{-1}(t))^{-n}\,\mathrm{d}t \geq c > 0.$$

Thus, in both cases

$$\begin{aligned} R^D f(x) &\geq \int_{D_\delta} G^D(x,y)f(y)\,\mathrm{d}y \\ &\geq c\int_{D_\delta} f(y)\,\mathrm{d}y \geq cV(\operatorname{diam}(D))\int_{D_\delta} f(y)\,\mathrm{d}y. \end{aligned} \tag{6.3.16}$$

Therefore, (6.3.14) follows from (6.3.15) and (6.3.16). $\square$

Remark 6.3.4. As a corollary of Lemma 6.3.3, we have

$$\|R^D f\|_{L^\infty(D)} \leq C\|f\|_{L^\infty(D)}.$$

Hence, the estimate (6.3.1) can be simplified into

$$\|R^D f\|_{C^V(\overline{B_{r/2}(x_0)})} \leq C\|f\|_{L^\infty(D)}. \tag{6.3.17}$$

We are now ready to prove Theorem 6.0.1, which will follow from Theorem 6.2.3 and the following proposition.

Proposition 6.3.5. *Assume $f \in L^\infty(D)$. Then, $R^D f \in C^V(\overline{D})$ and*

$$\|R^D f\|_{C^V(\overline{D})} \leq C\|f\|_{L^\infty(D)}$$

for some constant $C > 0$ depending on n, D, and Φ.

Proof. By (6.3.17) we have

$$|R^D f(x) - R^D f(y)| \leq C\|f\|_{L^\infty(D)} V(|x-y|) \tag{6.3.18}$$

for all x, y satisfying $|x-y| < d_D(x)/2$. We want to show that (6.3.18) holds, possibly with a larger constant C, for all $x, y \in D$.

Let (R_0, Λ) be the $C^{1,1}$ characteristics of D. Then D can be covered by finitely many balls $B(z_i, d_D(z_i)/2)$ with $z_i \in D$ and finitely many sets of the form $B(z_j^*, R_0) \cap D$ with $z_j^* \in \partial D$. Thus, it is enough to show that (6.3.18) holds for all $x, y \in B(z_j^*, R_0) \cap D$ possibly with a larger constant.

Let us fix $B(z_0^*, R_0) \cap D$ and assume that the outward normal vector at z_0 is $(0, \cdots, 0, -1)$. This is possible because the operator is rotation invariant. Let $x = (x', x_n)$ and $y = (y', y_n)$ be two points in $B(z_0^*, R_0) \cap D$, and let $r = |x-y|$. Let us define, for $k \geq 0$,

$$x^k = (x', x_n + \lambda^k r) \quad \text{and} \quad y^k = (y', y_n + \lambda^k r),$$

for some $1 - 2^{-1}(1+\Lambda^2)^{-1/2} \le \lambda < 1$. Since $(1+\Lambda^2)^{-1/2}(x^k)_n \le d_D(x^k)$, we have

$$|x^k - x^{k+1}| = \lambda^k(1-\lambda)r \le \frac{1}{2\sqrt{1+\Lambda^2}}(x^k)_n \le \frac{1}{2}d_D(x^k).$$

Thus, we have from (6.3.18) that

$$\begin{aligned}|R^D f(x^k) - R^D f(x^{k+1})| &\le C\|f\|_{L^\infty(D)} V(|x^k - x^{k+1}|)\\ &= C\|f\|_{L^\infty(D)} V(\lambda^k(1-\lambda)r),\end{aligned}$$

and similarly that $|R^D f(y^k) - R^D f(y^{k+1})| \le C\|f\|_{L^\infty(D)} V(\lambda^k(1-\lambda)r)$. Moreover, note that the distance from the line segment joining x^0 and y^0 to the boundary ∂D is larger than $r(1-\Lambda/2)$. Thus, this line can be split into finitely many line segments of length less than $r(1-\Lambda/2)/2$. The number of small line segments depends only on Λ. Therefore, we have $|R^D f(x^0) - R^D f(y^0)| \le C\|f\|_{L^\infty(D)} V(r)$ and hence

$$\begin{aligned}|R^D f(x) - R^D f(y)| &\le |R^D f(x^0) - R^D f(y^0)| + \sum_{k\ge 0} |R^D f(x^k) - R^D f(x^{k+1})|\\ &\quad + \sum_{k\ge 0} |R^D f(y^k) - R^D f(y^{k+1})|\\ &\le C\|f\|_{L^\infty(D)} \left(V(r) + \sum_{k\ge 0} V(\lambda^k(1-\lambda)r) \right)\\ &\le C\|f\|_{L^\infty(D)} \left(1 + C\sum_{k\ge 0} \left(\lambda^k(1-\lambda)\right)^{\sigma_1} \right) V(r)\\ &\le C\|f\|_{L^\infty(D)} V(r).\end{aligned}$$

We finish the proof by recalling that $r = |x-y|$. □

Remark 6.3.6. Every viscosity supersolution u to the problem

$$\begin{cases} Lu \le -1 & \text{in } D,\\ u = 0 & \text{in } \mathbb{R}^n \setminus D,\end{cases}$$

satisfies $u \geq cV(d_D)$ for some constant $c > 0$. Indeed, letting $v = R^D 1$, we have $Lv = -1$ by Theorem 6.2.3. Thus, by the comparison principle, we obtain $u \geq v = R^D 1$. The conclusion follows from the second assertion of Lemma 6.3.3. This provides the optimality of Theorem 6.0.2.

6.4 Barriers

In this section we follow an idea in [66, Section 2] and extend the results in [66, Section 2] to our setting. Since d_D is $C^{1,1}$ only near ∂D, we need to use the following "regularized version" of d_D, defined in [66, Definition 2.1].

Definition 6.4.1. We call $\psi : D \to (0, \infty)$ the *regularized version of* d_D if $\psi \in C^{1,1}(D)$ and it satisfies

$$\psi \asymp d_D, \quad \|\nabla \psi\| \leq C, \quad \text{and} \quad \|\nabla \psi(x) - \nabla \psi(y)\| \leq C|x - y|$$

for any $x, y \in D$, where the constant $C > 0$ depends only on D.

For $D = B_1$, there exists a regularized version of d_{B_1} which is C^2 and isotropic. Let us denote this function by Ψ. For any open ball $B_r = B_r(x_0)$, we will take the regularized version of d_{B_r} which is defined by $\Psi_r(x) := \Psi(\frac{x-x_0}{r})$. Then, Ψ_r satisfies

$$C^{-1} d_{B_r} \leq \Psi_r \leq C d_{B_r}, \quad \|\nabla \Psi_r\| \leq C \quad \text{and} \quad \|\nabla^2 \Psi_r\| \leq \frac{C}{r}. \tag{6.4.1}$$

Let us introduce a series of lemmas that will be used to construct a barrier for L. The first lemma is provided in [66, Lemma 2.4].

Lemma 6.4.2. *Assume that D is a bounded $C^{1,1}$ open set and let ψ be a regularized version of d_D. Then, for every $x \in \mathbb{R}^n$ and $x_0 \in D$, we have*

$$|\psi(x) - (\psi(x_0) + \nabla \psi(x_0) \cdot (x - x_0))_+| \leq C|x - x_0|^2, \tag{6.4.2}$$

for some constant $C > 0$ depending only on D. In particular, when $D = B_r$

and $\psi = \Psi_r$*, we have* (6.4.2) *with the constant* C *replaced by* C/r *for some* $C = C(n)$*.*

The next lemma is the counterpart of [66, Lemma 2.5].

Lemma 6.4.3. *Let* $U \subset \mathbb{R}^n$ *be a* $C^{1,1}$ *open set, which can be unbounded. There is a constant* $C > 0$ *such that for any* $x \in U$ *and* $0 < r \leq 1$*,*

$$\int_{U\cap(B_r(x)\setminus B_{\rho/2}(x))} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \leq C\frac{r}{V(r)}, \tag{6.4.3}$$

where $\rho = d_U(x)$*.*

Proof. Fix $x \in U$ and denote $B_r := B_r(x)$. Note that there is a constant $\kappa = \kappa(U) > 0$ such that the level set $\{d_U = t\}$ is $C^{1,1}$ for any $t \in (0, \kappa]$ since U is $C^{1,1}$. Without loss of generality we may assume $\kappa \leq r$ because κ can be arbitrarily small.

Since $B_R \cap \{d_U \geq \kappa\} = \emptyset$ for every $R \leq \kappa - \rho$, we have

$$\begin{aligned}
&\int_{(B_r\setminus B_{\rho/2})\cap\{d_U\geq\kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&= \int_{(B_r\setminus B_{\max\{\rho/2,\kappa-\rho\}})\cap\{d_U\geq\kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \int_{(B_r\setminus B_{2\kappa/3})\cap\{d_U\geq\kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)},
\end{aligned}$$

where we used $\rho/2 \vee (\kappa - \rho) \geq 2\kappa/3$ in the last inequality. Since

$$\kappa \leq d_U(y) \leq r + \kappa \leq 2r \quad \text{and} \quad \frac{2\kappa}{3} \leq |x-y| \leq r$$

for every $y \in (B_r \setminus B_{2\kappa/3}) \cap \{d_U \geq \kappa\}$, we obtain that

$$\begin{aligned}
&\int_{(B_r\setminus B_{2\kappa/3})\cap\{d_U\geq\kappa\}} \frac{V(d_D(y))}{d_D(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \int_{(B_r\setminus B_{2\kappa/3})\cap\{d_U\geq\kappa\}} \frac{V(2r)}{\kappa} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq CV(r)\int_0^r \frac{s}{\varphi(s)}\,\mathrm{d}s \leq C\frac{r^2}{V(r)} \leq C\frac{r}{V(r)},
\end{aligned} \tag{6.4.4}$$

with the help of Lemma 2.1.1 and (6.1.3). Thus, it suffices to estimate the integrand in the left-hand side of (6.4.3) in the set $(B_r \setminus B_{\rho/2}) \cap \{0 < d_U < \kappa\}$.

We utilize the following estimates on the Hausdorff measure in [67], that is, there is a constant $C(U) > 0$ such that that for every $x \in U$ and $t \in (0,\kappa)$,

$$\mathcal{H}^{n-1}(\{d_U = t\} \cap (B_{2^{-k+1}r} \setminus B_{2^{-k}r})) \leq C(2^{-k}r)^{n-1}, \tag{6.4.5}$$

which follows from the fact that the level set $\{d_U = t\}$ is $C^{1,1}$ for $t \in (0,\kappa)$.

Let us denote $\mathcal{B}_k := B_{2^{-k}r}$ for $k \geq 0$ and let $M \in \mathbb{N}$ be the integer satisfying $2^{-M}r \leq \rho/2 < 2^{-M+1}r$. Using that $|x-y| \geq 2^{-k}r$ for every $y \in \mathcal{B}_{k-1} \setminus \mathcal{B}_k$ and that φ is increasing, we have

$$\begin{aligned}
&\int_{(B_r\setminus B_{\rho/2})\cap\{0<d_U<\kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \sum_{k=1}^M \int_{(\mathcal{B}_{k-1}\setminus\mathcal{B}_k)\cap\{0<d_U<\kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2}\varphi(|x-y|)} \\
&\leq \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)} \int_{(\mathcal{B}_{k-1}\setminus\mathcal{B}_k)\cap\{0<d_U<\kappa\}} \frac{V(d_U(y))}{d_U(y)}\,\mathrm{d}y \\
&= \sum_{k=1}^M \frac{1}{(2^{-k}r)^{n-2}\varphi(2^{-k}r)} \int_{(\mathcal{B}_{k-1}\setminus\mathcal{B}_k)\cap\{0<d_U<\kappa\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)|\,\mathrm{d}y.
\end{aligned}$$

Here, we have used the fact that $|\nabla d_U(y)| = 1$ for $y \in \{0 < d_U < \kappa\}$ in the last equality (see [67]). For any $1 \leq k \leq M$ and $y \in \mathcal{B}_{k-1}$ we have $d_U(y) \leq 2^{-k+1}r + \rho \leq (2^{-k+1} + 2^{-M+2})r \leq 6 \cdot 2^{-k}r$, which implies $\mathcal{B}_{k-1} \subset$

$\{d_U < 6 \cdot 2^{-k} r\}$. Thus, combining this with the inequalities above, we have

$$\begin{aligned}
&\int_{(B_r \setminus B_{\rho/2}) \cap \{0 < d_U < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2} \varphi(|x-y|)} \\
&\leq \sum_{k=1}^{M} \frac{1}{(2^{-k} r)^{n-2} \varphi(2^{-k} r)} \int_{(\mathcal{B}_{k-1} \setminus \mathcal{B}_k) \cap \{0 < d_U < 6 \cdot 2^{-k} r\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)| \, \mathrm{d}y.
\end{aligned} \tag{6.4.6}$$

Plugging $u(y) = d_U(y)$ and $g(y) = \frac{V(d_U(y))}{d_U(y)}$ into the following coarea formula

$$\int_D g(y) |\nabla u(y)| \, \mathrm{d}y = \int_{-\infty}^{\infty} \left(\int_{u^{-1}(t)} g(y) \, \mathrm{d}\mathcal{H}_{n-1}(y) \right) \mathrm{d}t,$$

and using (6.4.5) and (6.1.7), we obtain

$$\begin{aligned}
&\sum_{k=1}^{M} \frac{1}{(2^{-k} r)^{n-2} \varphi(2^{-k} r)} \int_{(\mathcal{B}_{k-1} \setminus \mathcal{B}_k) \cap \{0 < d_U < 6 \cdot 2^{-k} r\}} \frac{V(d_U(y))}{d_U(y)} |\nabla d_U(y)| \, \mathrm{d}y \\
&= \sum_{k=1}^{M} \frac{1}{(2^{-k} r)^{n-2} \varphi(2^{-k} r)} \int_0^{6 \cdot 2^{-k} r} \int_{(\mathcal{B}_{k-1} \setminus \mathcal{B}_k) \cap \{d = t\}} \frac{V(t)}{t} \, \mathrm{d}\mathcal{H}^{n-1}(y) \, \mathrm{d}t \\
&\leq \sum_{k=1}^{M} \frac{1}{(2^{-k} r)^{n-2} \varphi(2^{-k} r)} \int_0^{6 \cdot 2^{-k} r} C (2^{-k} r)^{n-1} \frac{V(t)}{t} \, \mathrm{d}t \\
&= C \sum_{k=1}^{M} \frac{2^{-k} r}{\varphi(2^{-k} r)} \int_0^{6 \cdot 2^{-k} r} \frac{V(t)}{t} \, \mathrm{d}t \leq C \sum_{k=1}^{M} \frac{2^{-k} r}{\varphi(2^{-k} r)} V(6 \cdot 2^{-k} r).
\end{aligned} \tag{6.4.7}$$

It is now standard to estimate

$$\begin{aligned}
\sum_{k=1}^{M} \frac{2^{-k} r}{\varphi(2^{-k} r)} V(6 \cdot 2^{-k} r) &\leq C \sum_{k=1}^{M} \frac{2^{-k} r}{V(2^{-k} r)} = C \sum_{k=1}^{M} \int_{2^{-k} r}^{2^{-k+1} r} \frac{\mathrm{d}s}{V(2^{-k} r)} \\
&\leq C \int_0^r \frac{\mathrm{d}s}{V(s)} \leq C \frac{r}{V(r)}.
\end{aligned} \tag{6.4.8}$$

Therefore, it follows from (6.4.6), (6.4.7), and (6.4.8), that

$$\int_{(B_r \setminus B_{\rho/2}) \cap \{d < \kappa\}} \frac{V(d_U(y))}{d_U(y)} \frac{\mathrm{d}y}{|x-y|^{n-2} \varphi(|x-y|)} \leq C \frac{r}{V(r)}.$$

This and (6.4.4) finish the proof. $\square$

With the help of lemmas above, we are now ready to show that $V(\psi)$ acts as a barrier of L on D.

Proposition 6.4.4. *Assume that D is a bounded $C^{1,1}$ open set and let ψ be a regularized version of d_D. There is a constant $C > 0$ such that*

$$|L(V(\psi))| \leq C \quad \text{in } D, \tag{6.4.9}$$

where V is the renewal function with respect to Φ. In particular, when $D = B_r$ and $\psi = \Psi_r$, there is a constant $C > 0$, independent of r, such that

$$|L(V(\psi))| \leq \frac{C}{V(r)} \quad \text{in } B_r. \tag{6.4.10}$$

Proof. The proof is mainly motivated by [66, Proposition 2.3]. Let us provide the proof of (6.4.10) only, since the proof of (6.4.9) is very similar. Fix $x_0 \in B_r$ and let $\rho := d_{B_r}(x_0)$. We first prove (6.4.10) for the case $\rho \geq \kappa r > 0$ with a small constant κ which will be chosen later. In this case, we have

$$\begin{aligned} |L(V(\psi))(x_0)| &= \left| \frac{1}{2} \int_{\mathbb{R}^n} \delta(V(\psi), x_0, y) \frac{J(1)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \right| \\ &\leq C \int_{B_{\kappa r/2}} \|\nabla^2 (V(\psi))(x^*)\| \frac{\mathrm{d}y}{|y|^{n-2} \varphi(|y|)} \\ &\quad + C \int_{B^c_{\kappa r/2}} \frac{|\delta(V(\psi), x_0, y)|}{|y|^n \varphi(|y|)} \,\mathrm{d}y =: I_1 + I_2, \end{aligned} \tag{6.4.11}$$

where x^* is some point on the line segment between $x_0 + y$ and $x_0 - y$, so that $d_{B_r}(x^*) \geq \kappa r/2$ when $y \in B_{\kappa r/2}$. Using (6.1.4), (6.4.1), and Lemma 6.1.3, we have

$$\begin{aligned} \|\nabla^2 (V(\psi))(x^*)\| &\leq |V''(\psi(x^*))| \|\nabla \psi(x^*)\|^2 + |V'(\psi(x^*))| \|\nabla^2 \psi(x^*)\| \\ &\leq C \frac{V(r)}{r^2}. \end{aligned}$$

Thus, the first term in the right-hand side of (6.4.11) is estimated by

$$I_1 \leq C\frac{V(r)}{r^2}\int_{B_{\kappa r/2}} \frac{\mathrm{d}y}{|y|^{n-2}\varphi(|y|)} \leq C\frac{V(r)}{r^2}\int_0^{\kappa r/2} \frac{s}{\varphi(s)}\,\mathrm{d}s \leq \frac{C}{V(r)}.$$

For the second term, using (6.4.1), we have $|\delta(V(\psi), x_0, y)| \leq 4V(Cr) \leq C$. Therefore, we have

$$I_2 \leq CV(r)\int_{B^c_{\kappa r/2}} \frac{\mathrm{d}y}{|y|^{n}\varphi(|y|)} \leq CV(r)\int_{\kappa r/2}^{\infty} \frac{\mathrm{d}s}{s\varphi(s)} \leq \frac{C}{V(r)}.$$

We have proved (6.4.10) for the case $\rho \geq \kappa r$.

Let us next consider the case $\rho < \kappa r$. We know from (6.1.2) that the function l defined by $l(x) := (\psi(x_0)+\nabla\psi(x_0)\cdot(x-x_0))_+$ satisfies $L(V(l)) = 0$ on $\{l > 0\}$. Note that $\psi(x_0) = l(x_0)$ and $\nabla\psi(x_0) = \nabla l(x_0)$. Moreover, by Lemma 6.4.2 we have

$$|\psi(x) - l(x)| \leq \frac{C}{r}|x-x_0|^2. \tag{6.4.12}$$

For any $a < b$, there exists $a_* \in [a,b]$ satisfying $|V(a)-V(b)| = |a-b|V'(a_*)$. Using Lemma 6.1.3, we obtain

$$|V(a) - V(b)| = |a-b|V'(a_*) \leq C|a-b|\frac{V(a_*)}{a_*} \leq C|a-b|\frac{V(a)}{a}.$$

Therefore, for any $a, b > 0$ we have

$$|V(a) - V(b)| \leq C|a-b|\left(\frac{V(a)}{a} + \frac{V(b)}{b}\right). \tag{6.4.13}$$

By (6.4.13) and (6.4.12) we obtain that for any $x \in B_r(x_0)$,

$$
\begin{aligned}
&|V(\psi(x)) - V(l(x))| \\
&\leq \frac{C}{r}|x - x_0|^2 \left(\frac{V(\psi(x))}{\psi(x)} \mathbf{1}_{\{\psi(x)>0\}} + \frac{V(l(x))}{l(x)} \mathbf{1}_{\{l(x)>0\}} \right) \\
&\leq \frac{C}{r}|x - x_0|^2 \left(\frac{V(d_{B_r}(x))}{d_{B_r}(x)} \mathbf{1}_{\{d_{B_r}(x)>0\}} + \frac{V(l(x))}{l(x)} \mathbf{1}_{\{l(x)>0\}} \right),
\end{aligned}
\tag{6.4.14}
$$

where we have used the comparability of ψ and d_{B_r}, and the weak scaling condition (6.1.4).

On the other hand, if $x \in B_{\rho/2}(x_0)$ with $\rho \leq \kappa r$, then by taking κ sufficiently small, we see that $\psi(x)$ and $l(x)$ are comparable to ρ. Therefore, since $|V(\psi(x)) - V(l(x))| = |\psi(x) - l(x)|V'(z)$ for some value z in between $\psi(x)$ and $l(x)$, we have

$$
\begin{aligned}
|V(\psi(x)) - V(l(x))| &= |\psi(x) - l(x)|V'(z) \\
&\leq \frac{C}{|x - x_0|^2} \frac{V(z)}{z} \leq \frac{C}{r}|x - x_0|^2 \frac{V(\rho)}{\rho},
\end{aligned}
\tag{6.4.15}
$$

by using (6.1.10) and (6.1.4).

For $x \in \mathbb{R}^n \setminus B_r(x_0)$, we have

$$
\begin{aligned}
V(l(x)) &= V((\psi(x_0) + \nabla\psi(x_0) \cdot (x - x_0))_+) \\
&\leq V(C\rho + C|x - x_0|) \leq CV(|x - x_0|)
\end{aligned}
$$

and

$$
V(\psi(x)) \leq V(Cr) \leq CV(|x - x_0|).
$$

Thus, we obtain that

$$
|V(\psi) - V(l)|(x) \leq CV(|x - x_0|). \tag{6.4.16}
$$

Therefore, by taking $x = y + x_0$ in (6.4.14), (6.4.15), and (6.4.16), we have

$$
\begin{aligned}
&|V(\psi) - V(l)|(y + x_0) \\
&\leq C \begin{cases} \frac{1}{r}\frac{V(\rho)}{\rho}|y|^2, & |y| < \frac{\rho}{2}, \\ \frac{|y|^2}{r}\left(\frac{V(d_{B_r}(x_0+y))}{d_{B_r}(x_0+y)}\mathbf{1}_{\{d_{B_r}(x_0+y)>0\}} + \frac{V(l(x_0+y))}{l(x_0+y)}\mathbf{1}_{\{l(x_0+y)>0\}}\right), & \frac{\rho}{2} \leq |y| < r, \\ V(|y|), & r \leq |y|. \end{cases}
\end{aligned}
$$

Recalling that $L(V(l))(x_0) = 0$ and $\psi(x_0) = l(x_0)$, we find that

$$
\begin{aligned}
|L(V(\psi))(x_0)| &= |L\left(V(\psi(\cdot)) - V(l(\cdot))\right)(x_0)| \\
&\leq \int_{\mathbb{R}^n} |V(\psi) - V(l)|(x_0 + y)\frac{J(1)}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\
&\leq \frac{C}{r}\frac{V(\rho)}{\rho}\int_{B_{\rho/2}} \frac{|y|^2}{|y|^n\varphi(|y|)}\,\mathrm{d}y + C\int_{\mathbb{R}^n\setminus B_r} \frac{V(|y|)}{|y|^n\varphi(|y|)}\,\mathrm{d}y \\
&\quad + C\int_{B_r\setminus B_{\rho/2}} \frac{|y|^2}{r}\frac{V(d_{B_r}(x_0+y))}{d_{B_r}(x_0+y)}\mathbf{1}_{\{d_{B_r}(x_0+y)>0\}}\frac{\mathrm{d}y}{|y|^n\varphi(|y|)} \\
&\quad + C\int_{B_r\setminus B_{\rho/2}} \frac{|y|^2}{r}\frac{V(l(x_0+y))}{l(x_0+y)}\mathbf{1}_{\{l(x_0+y)>0\}}\frac{\mathrm{d}y}{|y|^n\varphi(|y|)} \\
&= I_3 + I_4 + I_5 + I_6.
\end{aligned}
$$

It is now very standard to estimate I_3 and I_4; by using Lemma 2.1.1, (6.0.4), (6.1.3), (6.1.4), and (6.1.8), we have

$$
I_3 \leq \frac{C}{r}\frac{V(\rho)}{\rho}\frac{\rho^2}{\varphi(\rho)} \leq \frac{C}{V(r)}\left(\frac{\rho}{r}\frac{V(r)}{V(\rho)}\right) \leq \frac{C}{V(r)}
$$

and

$$
I_4 \leq C\int_r^\infty \frac{V(s)}{s\varphi(s)}\,\mathrm{d}s \leq \frac{C}{V(r)}.
$$

In order to estimate I_5 and I_6, let $U_1 = B_r$ and $U_2 = \{l > 0\}$. We observe that for $z \in U_2$,

$$
\left|\frac{l(z)}{d_H(z)}\right| = \|\nabla\psi(x_0)\| \leq C.
$$

Thus, by (6.1.4) we have

$$\frac{V(l(z))}{l(z)} \leq C\frac{V(Cd_{U_2}(z))}{d_{U_2}(z)} \leq C\frac{V(d_{U_2}(z))}{d_{U_2}(z)}.$$

Therefore, we apply Lemma 6.4.3 to I_5 and I_6 with U_1 and U_2, respectively, to obtain that

$$I_5 + I_6 \leq \frac{C}{r}\sum_{i=1}^{2}\int_{U_i\cap\left(B_r(x_0)\setminus B_{\rho/2}(x_0)\right)} \frac{V(d_{U_i}(z))}{d_{U_i}(z)} \frac{\mathrm{d}z}{|z-x_0|^{n-2}\varphi(|z-x_0|)} \leq \frac{C}{V(r)}.$$

Combining estimates of I_3–I_6, we arrive at

$$|L(V(\psi))(x_0)| \leq \frac{C}{V(r)},$$

which finishes the proof. $\square$

Recall that the domain of the infinitesimal generator A was defined by

$$\mathcal{D} = \mathcal{D}(D) = \{u \in C_0(D) : Au \in C(D)\}.$$

However, it is uncertain whether $V(\psi) \in \mathcal{D}(D)$ since $A(V(\psi))$ is not continuous in general. Therefore, we construct a larger domain of generator that contains $V(\psi)$. For a given $C^{1,1}$ bounded open set D and open subset U of D, we define

$$\mathcal{F} := \mathcal{F}(D,U) := \{u \in C_0(D) : Au \in L^\infty(U)\},$$

and denote $\mathcal{F}(D) := \mathcal{F}(D,D)$. Clearly, we have $\mathcal{F}(D,U_2) \subset \mathcal{F}(D,U_1)$ for any $U_1 \subset U_2$. Let us check that the new domain contains a function $V(\psi)$.

Lemma 6.4.5. *Let ψ be the regularized version of d_D. Then, $A(V(\psi)) = L(V(\psi))$ in D. Moreover, $V(\psi) \in \mathcal{F}(D)$.*

Proof. Let $u \in C_0(D)$ be a twice-differentiable function in D and assume that D^2u is bounded in some $U \subset\subset D$. We first claim that

$$Lu(x) = Au(x) \quad \text{for any } x \in U.$$

Indeed, fix $x \in U$ and let $r_x > 0$ be a constant satisfying $B = B(x, r_x) \subset U$. Without loss of generality we may assume $r_x \le 1$. Note that there exists a constant $C > 0$ such that $2|u| + r_x^2 \|\nabla^2 u\| \le C$ in U. Then we have

$$\begin{aligned} Au(x) &= \lim_{t \searrow 0} \frac{P_t u(x) - u(x)}{t} \\ &= \lim_{t \searrow 0} \frac{1}{t} \left(\int_{\mathbb{R}^n} u(x+y) p(t, |y|) \, \mathrm{d}y - u(x) \right) \\ &= \lim_{t \searrow 0} \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{2} \frac{p(t, |y|)}{t} \, \mathrm{d}y. \end{aligned}$$

Since $\frac{p(t,r)}{t} \le CJ(r)$ for $t > 0$ and $r > 0$ for some $C > 0$, we have

$$\begin{aligned} \left| \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{2} \frac{p(t, |y|)}{t} \, \mathrm{d}y \right| &\le C \int_B \frac{|y|^2}{r_x^2} \frac{p(t, |y|)}{t} \, \mathrm{d}y + C \int_{\mathbb{R}^n \setminus B} \frac{p(t, |y|)}{t} \, \mathrm{d}y \\ &\le C \int_{\mathbb{R}^n} \left(\frac{|y|^2}{r_x^2} \wedge 1 \right) J(|y|) \, \mathrm{d}y < +\infty \end{aligned}$$

for any $t > 0$. Thus, by using the dominate convergence theorem and using $\lim_{t \searrow 0} \frac{p(t,r)}{t} = J(r)$ we obtain that

$$Au(x) = \frac{1}{2} \int_{\mathbb{R}^n} \delta(u, x, y) J(|y|) \, \mathrm{d}y = Lu(x),$$

which concludes the claim.

We know from Lemma 6.1.3 that $V(\psi) \in C_0(D)$ is twice-differentiable and $D^2(V(\psi))$ is locally bounded on D. Therefore, by the claim we have $L(V(\psi)) = A(V(\psi))$ in D. The second assertion follows immediately from (6.4.9). □

We are now ready to construct a barrier with respect to the generator A.

Lemma 6.4.6. *There is a radial function $w = w_r \in \mathcal{F}(B_{4r})$ such that*

$$\begin{cases} Aw \geq 0 & \text{in } B_{4r} \setminus B_r, \\ w \leq V(r) & \text{in } B_r, \\ w \geq CV(4r - |x|) & \text{in } B_{4r} \setminus B_r, \\ w \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4r}, \end{cases}$$

for some constant $C > 0$ independent of r.

Proof. Let $\Psi = \Psi_{4r}$ be the regularized version of $d_{B_{4r}}$ in (6.4.1) and choose a function $\eta \in C_c^\infty(B_1)$ satisfying $\eta \in [0,1]$ and $\eta \equiv 1$ on $B_{1/2}$. Define $\eta_r(x) := V(r)\eta(x/r) \in C_c^\infty(B_r)$. It is clear that $\eta_r \in \mathcal{F}(B_{4r})$. For $x \in B_{4r} \setminus B_r$, by (6.1.3) and (6.1.4) we have

$$\begin{aligned} A\eta_r(x) = L\eta_r(x) &= \int_{\mathbb{R}^n} \eta_r(x+y) \frac{J(1)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \\ &\geq \int_{B(-x,r/2)} \frac{V(r)J(1)}{|y|^n \varphi(|y|)} \,\mathrm{d}y \geq \frac{c}{V(4r)}. \end{aligned}$$

Define a function $\tilde{w}_r$ by

$$\tilde{w}_r = \frac{c}{C_1} V(\Psi) + \eta_r,$$

where C_1 is the constant in (6.4.10). Then by Lemma 6.4.5, we have $\tilde{w}_r \in \mathcal{F}(B_{4r})$. Moreover, for $x \in B_{4r} \setminus B_r$ we obtain

$$A\tilde{w}_r(x) \geq -\frac{c}{C_1} |LV(\Psi)(x)| + A\eta_r(x) \geq -\frac{c}{V(4r)} + \frac{c}{V(4r)} = 0$$

and

$$\tilde{w}_r(x) = \frac{c}{C_1} V(\Psi(x)) \geq CV(d_D(x)) = CV(4r - |x|).$$

For $x \in B_r$,

$$\tilde{w}_r(x) \leq \frac{c}{C_1} V(4Cr) + V(r) \leq C_2 V(r)$$

by (6.4.1) and (6.1.4). Define $w_r(x) := \frac{1}{C_2}\tilde{w}_r(x)$, then w_r satisfies all assertions in Lemma 6.4.6. $\square$

For local operators, Hölder regularity of solutions follow immediately from the Harnack inequality. However for nonlocal operators, as Silvestre mentioned in [76], this is not the case because the nonnegativity of the function u is required in the whole space $\mathbb{R}^n$. The Harnack inequality for viscosity solutions established in Theorem 3.0.2 and the barrier function constructed in Lemma 6.4.6 will play a key role in the proof of Theorem 6.0.2. We close this section with the following probabilistic version of the maximum principle that will be needed in the barrier argument.

Lemma 6.4.7 (Maximum principle)**.** *Let D be a bounded $C^{1,1}$ open set and U be an open subset of D. If $u \in \mathcal{F}(D, U)$ satisfies $Au = 0$ a.e. in U and $u \geq 0$ in $\mathbb{R}^n \setminus U$, then $u \geq 0$ in $\mathbb{R}^n$.*

Proof. Suppose that there exists a point $x \in U$ such that $u(x) < 0$. Since $u \in C_0(D)$, the set $U_- := \{x \in \mathbb{R}^n : u(x) < 0\}$ is open and bounded with a positive Lebesgue measure. For any $t > 0$, we have

$$\begin{aligned}\int_{U_-} (P_t u(x) - u(x))\,\mathrm{d}x &= \int_{U_-}\int_{\mathbb{R}^n} u(y)p(t,|x-y|)\,\mathrm{d}y\,\mathrm{d}x - \int_{U_-} u(y)\,\mathrm{d}y\\ &= \int_{\mathbb{R}^n} u(y)\int_{U_-} p(t,|x-y|)\,\mathrm{d}x\,\mathrm{d}y - \int_{U_-} u(y)\,\mathrm{d}y\\ &\geq \int_{U_-} u(y)\left(\int_{U_-} p(t,|x-y|)\,\mathrm{d}x - 1\right)\mathrm{d}y.\end{aligned}$$

Let $R := \operatorname{diam}(U_-) < +\infty$, then for any $y \in U_- \subset B_R(y)$,

$$\begin{aligned}\frac{1}{t}\left(1 - \int_{U_-} p(t,|x-y|)\,\mathrm{d}x\right) &\geq \frac{1}{t}\left(1 - \int_{B_R(y)} p(t,|x-y|)\,\mathrm{d}x\right)\\ &= \frac{1}{t}\left(1 - \mathbb{P}^y(X_t \in B_R(y))\right) = \frac{\mathbb{P}^0(|X_t| \geq R)}{t}.\end{aligned}$$

Using the heat kernel estimates in [9, Theorem 21], we have

$$p(t,r) \asymp \left(\varphi^{-1}(t)^{-n} \wedge \frac{t}{r^n\varphi(r)}\right) \quad \text{for } (t,r) \in (0,1] \times \mathbb{R}_+.$$

Since $\frac{t}{r^n\varphi(r)} \le \varphi^{-1}(t)^{-n}$ for $t \le \varphi(r)$, there exists $\varepsilon = \varepsilon(R) > 0$ such that

$$\frac{\mathbb{P}^0(|X_t| \ge R)}{t} \ge \frac{1}{t}\int_{R\le|z|\le 2R} p(t,|z|)\,\mathrm{d}z \ge c\int_R^{2R} \frac{\mathrm{d}r}{r\varphi(r)} \ge \varepsilon$$

for all $t \in (0, \varphi(R)]$. Therefore, we obtain

$$\int_{U_-} \frac{P_t u(x) - u(x)}{t}\,\mathrm{d}x \ge -\varepsilon\int_{U_-} u(y)\,\mathrm{d}y \quad \text{for all } t \in (0, \varphi(R)].$$

Letting $t \to 0$, we arrive at

$$0 = \int_{U_-} Au(x)\,\mathrm{d}x = \lim_{t\to 0}\int_{U_-} \frac{P_t u(x) - u(x)}{t}\,\mathrm{d}x \ge -\varepsilon\int_{U_-} u(y)\,\mathrm{d}y > 0,$$

which is a contradiction. Therefore, $u \ge 0$ in $\mathbb{R}^n$. □

6.5 Higher Order Regularity up to Boundary

In this section we will prove Theorem 6.0.2. More precisely, we prove the Hölder regularity for the function $u/V(d_D)$ up to the boundary of D. We will control the oscillation of this function using the Harnack inequality, the maximum principle, and the barrier function constructed in Lemma 6.4.6.

Let us adopt notations in [63, Definition 3.3]. Let $\kappa > 0$ be a fixed small constant and let $\kappa' = 1/2 + 2\kappa$. Given $x_0 \in \partial D$ and $r > 0$, define

$$D_r = D_r(x_0) = B(x_0, r) \cap D$$

and

$$D^+_{\kappa' r} = D^+_{\kappa' r}(x_0) = B(x_0, \kappa' r) \cap \{x \in D : -x\cdot\nu(x_0) \ge 2\kappa r\},$$

where $\nu(x_0)$ is the unit outward normal at x_0. Since D is a bounded $C^{1,1}$ open set, there exists $\rho_0 > 0$ such that for each $x_0 \in \partial D$ and $r \le \rho_0$, there exists an orthonormal system CS_{x_0} with its origin at x_0 and a $C^{1,1}$ function

$\Psi : \mathbb{R}^{n-1} \to \mathbb{R}$ satisfying $\Psi(\tilde{0}) = 0, \nabla_{CS_{x_0}} \Psi(\tilde{0}) = 0, \|\Psi\|_{C^{1,1}} \leq \kappa$, and

$$\{y = (\tilde{y}, y_n) \text{ in } CS_{x_0} : |\tilde{y}| < 2r, \Psi(\tilde{y}) < y_n < 2r\} \subset D.$$

Then we have

$$B(y, \kappa r) \subset D_r(x_0) \text{ for all } y \in D^+_{\kappa' r}(x_0), \tag{6.5.1}$$

and we can take a $C^{1,1}$ subdomain $D^{1,1}_r$ satisfying $D_r \subset D^{1,1}_r \subset D_{2r}$ and

$$\operatorname{dist}(y, \partial D^{1,1}_r) = d_D(y)$$

for all $y \in D_r$. Since D_r is not $C^{1,1}$ in general, we will use this subdomain instead of D_r.

Since D is bounded and $C^{1,1}$ again, we may assume that for each $x_0 \in \partial D$ and $r \leq \rho_0$,

$$B(y^* - 4\kappa r \nu(y^*), 4\kappa r) \subset D_r(x_0) \text{ and } B(y^* - 4\kappa r \nu(y^*), \kappa r) \subset D^+_{\kappa' r}(x_0) \tag{6.5.2}$$

for all $y \in D_{r/2}(x_0)$, where $y^* \in \partial D$ is the unique boundary point satisfying $|y - y^*| = d_D(y)$.

The following oscillation lemma is the key lemma to prove Theorem 6.0.2.

Lemma 6.5.1 (Oscillation lemma). *Assume $f \in C(D)$ and let $u \in \mathcal{D}$ be the viscosity solution of* (6.0.1)*. Then there exist constants $\gamma \in (0, 1)$ and $C > 0$, depending only on n, D, and Φ, such that*

$$\sup_{D_r(x_0)} \frac{u}{V(d_D)} - \inf_{D_r(x_0)} \frac{u}{V(d_D)} \leq C V(r)^\gamma \|f\|_{L^\infty(D)}$$

for any $x_0 \in \partial D$ and $r > 0$.

The first step towards Lemma 6.5.1 is the Harnack inequality for the function $u/V(d_D)$.

Lemma 6.5.2 (Harnack inequality). *Let $f \in L^\infty(D^{1,1}_r)$ and let u be a vis-*

cosity solution of $Lu = f$ *in* $D_r^{1,1}$*. Then for any* $x_0 \in \partial D$ *and* $r \leq \rho_0$*,*

$$\sup_{D^+_{\kappa' r}(x_0)} \frac{u}{V(d_D)} \leq C \left(\inf_{D^+_{\kappa' r}(x_0)} \frac{u}{V(d_D)} + \|f\|_{L^\infty(D_r^{1,1})} V(r) \right), \tag{6.5.3}$$

for some constant $C > 0$.

Proof. We are going to utilize Theorem 3.0.2 in order to prove (6.5.3). For each $y \in D^+_{\kappa' r}$, we have $B(y, \kappa r) \subset D_r^{1,1}$ by (6.5.1). Thus, we may cover $D^+_{\kappa' r}$ by finitely many balls $B(y_i, \kappa r/2)$, with the number of balls independent of r. By applying Theorem 3.0.2 to u on each ball $B(y_i, \kappa r/2)$, we obtain

$$\begin{aligned} \sup_{B(y_i, \kappa r/2)} v &\leq C \left(\inf_{B(y_i, \kappa r/2)} u + \frac{(\kappa r/2)^2}{\underline{C}_\varphi(\kappa r/2)} \|f\|_{L^\infty(D_r^{1,1})} \right) \\ &\leq C \left(\inf_{B(y_i, \kappa r/2)} u + \|f\|_{L^\infty(D_r^{1,1})} \varphi(r) \right), \end{aligned}$$

where we used Lemma 2.1.1 and the weak scaling condition (6.0.4) in the last inequality. If $x \in B(y_i, \kappa r/2)$, we have $\kappa r/2 \leq d_D(x) \leq r/2 + 5\kappa r/2$. Thus, using (6.1.4) and (6.1.3) we obtain

$$\begin{aligned} \sup_{B(y_i, \kappa r/2)} \frac{u}{V(d_D)} &\leq \sup_{B(y_i, \kappa r/2)} \frac{u}{V(\kappa r/2)} \\ &\leq C \left(\inf_{B(y_i, \kappa r/2)} \frac{u}{V(r/2 + 5\kappa r/2)} + \|f\|_{L^\infty(D_r^{1,1})} \frac{\varphi(r)}{V(r)} \right) \\ &\leq C \left(\inf_{B(y_i, \kappa r/2)} \frac{u}{V(d_D)} + \|f\|_{L^\infty(D_r^{1,1})} V(r) \right). \end{aligned}$$

Therefore, the standard chaining argument proves (6.5.3), possibly with a larger constant. □

The next lemma provides the link between the sets $D^+_{\kappa' r}$ and $D_{r/2}$. Since we are going to use the barrier function w constructed in Lemma 6.4.6, we will consider a function u in a larger domain $\mathcal{F}$.

Lemma 6.5.3. *Let* $r \leq \rho_0$ *and* $x_0 \in \partial D$*. If* $u \in \mathcal{F}(D, D_r^{1,1})$ *is nonnegative,* *then*

$$\inf_{D^+_{\kappa' r}(x_0)} \frac{u}{V(d_D)} \le C \left(\inf_{D_{r/2}(x_0)} \frac{u}{V(d_D)} + \|Au\|_{L^\infty(D_r^{1,1})} V(r) \right),$$

for some constant $C > 0$.

Proof. Let us write $u = u_1 + u_2$ with $u_1 = u + R^{D_r^{1,1}} Au$ and $u_2 = -R^{D_r^{1,1}} Au$. Note that by Lemma 6.3.3 and (6.5.1), we have

$$\begin{aligned} |u_2(x)| &\le C\|Au\|_{L^\infty(D_r^{1,1})} V(\operatorname{diam}(D_r^{1,1})) V(\operatorname{dist}(x, \partial D_r^{1,1})) \\ &\le C\|Au\|_{L^\infty(D_r^{1,1})} V(r) V(d_D(x)). \end{aligned}$$

Therefore, it only remains to show that

$$\inf_{D^+_{\kappa' r}} \frac{u_1}{V(d_D)} \le C \inf_{D_{r/2}} \frac{u_1}{V(d_D)}. \tag{6.5.4}$$

Let

$$m := \inf_{D^+_{\kappa' r}} \frac{u_1}{V(d_D)}.$$

Since u_1 is a nonnegative solution to

$$\begin{cases} Au_1 = 0 & \text{a.e. in } D_r^{1,1}, \\ u_1 = u & \text{in } \mathbb{R}^n \setminus D_r^{1,1}, \end{cases}$$

we find that $u_1 \ge 0$ by Lemma 6.4.7, and hence $m \ge 0$.

For $y \in D_{r/2}$, we have either $y \in D^+_{\kappa' r}$ or $d_D(y) < 4\kappa r$ by (6.5.2). If $y \in D^+_{\kappa' r}$, then clearly we have

$$m \le \frac{u_1(y)}{V(d_D(y))}. \tag{6.5.5}$$

If $d_D(y) < 4\kappa r$, let y^* be the closest point to y on $\partial D_r^{1,1}$ and let $\tilde{y} = y^* - 4\kappa r \nu(y^*)$. By (6.5.2), we have $B_{4\kappa r}(\tilde{y}) \subset D_r$ and $B_{\kappa r}(\tilde{y}) \subset D^+_{\kappa' r}$. Let us consider a function $w \in \mathcal{F}(B_{4\kappa r}(\tilde{y})) \subset \mathcal{F}(D, B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}))$ satisfying

$$\begin{cases} Aw \geq 0 & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \leq V(\kappa r) & \text{in } B_{\kappa r}(\tilde{y}), \\ w \geq cV(4\kappa r - |x - \tilde{y}|) & \text{in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ w \equiv 0 & \text{in } \mathbb{R}^n \setminus B_{4\kappa r}(\tilde{y}), \end{cases}$$

which can be obtained by translating the function in Lemma 6.4.6. Since $Au_1 = 0$ a.e. in $B_{4\kappa r}(\tilde{y})$, we have

$$\begin{cases} Au_1 = 0 \leq A(mw) & \text{a.e. in } B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y}), \\ u_1 \geq mV(d_D) \geq mw & \text{in } B_{\kappa r}(\tilde{y}), \\ u_1 \geq 0 = mw & \text{in } \mathbb{R}^n \setminus B_{4\kappa r}(\tilde{y}). \end{cases}$$

Therefore, by Lemma 6.4.7, we obtain $u_1 \geq mw$ in $\mathbb{R}^n$. In particular, for $y \in B_{4\kappa r}(\tilde{y}) \setminus B_{\kappa r}(\tilde{y})$, we have $u_1(y) \geq cmV(4\kappa r - |y - \tilde{y}|) = cmV(d_D(y))$. This, together with (6.5.5), proves (6.5.4). $\square$

We prove Lemma 6.5.1 by using Lemma 6.5.2 and Lemma 6.5.3.

Proof of Lemma 6.5.1. As a consequence of Remark 6.3.4, we may assume $\|f\|_{L^\infty(D)} \leq 1$ and $\|u\|_{C(D)} = \|R^D f\|_{C(D)} \leq C$ without loss of generality by dividing $\|f\|_{L^\infty(D)}$ on both sides of (6.0.1) if necessary. Fix $x_0 \in \partial D$. We will prove that there are constants $C_1 > 0, \rho_1 \in (0, \rho_0/16]$, and $\gamma \in (0,1)$, and monotone sequences $(m_k)_{k\geq 0}$ and $(M_k)_{k\geq 0}$ such that $M_k - m_k = V(r_{k+1}/2)^\gamma$,

$$-V(\rho_1/16) \leq m_k \leq m_{k+1} < M_{k+1} \leq M_k \leq V(\rho_1/16),$$

and

$$m_k \leq \frac{u}{C_1 V(d_D)} \leq M_k \quad \text{in } D_{r_k} = D_{r_k}(x_0)$$

for all $k \geq 0$, where $r_k = \rho_1 8^{-k}$. If we have such constants and sequences, then for any $0 < r \le \rho_1$ we have $k \ge 0$ satisfying $r \in (r_{k+1}, r_k]$ and

$$\begin{aligned}\sup_{D_r} \frac{u}{V(d_D)} - \inf_{D_r} \frac{u}{V(d_D)} &\le \sup_{D_{r_k}} \frac{u}{V(d_D)} - \inf_{D_{r_k}} \frac{u}{V(d_D)} \\ &\le C_1(M_k - m_k) = C_1 V(r_{k+1}/2)^\gamma \le C_1 V(r)^\gamma.\end{aligned}$$

Moreover, for any $r > \rho_1$ we have

$$\sup_{D_r} \frac{u}{V(d_D)} - \inf_{D_r} \frac{u}{V(d_D)} \le C \le CV(\rho_1)^\gamma \le CV(r)^\gamma$$

by Lemma 6.3.3. Two inequalities above conclude the lemma so it suffices to construct such constants and sequences.

Let us use the induction on k. The case $k = 0$ follows from Lemma 6.3.3 provided we take C_1 large enough. The constants ρ_1 and γ will be chosen later. Assume that we have sequences up to m_k and M_k. Let ψ be the regularized version of d_D. We may assume that $\psi = d_D$ in $\{d_D(x) \le \rho_1\}$. Define

$$u_k = V(\psi)\left(\frac{u}{C_1 V(\psi)} - m_k\right) = \frac{1}{C_1}u - m_k V(\psi)$$

in $\mathbb{R}^n$. Note that $u_k \in \mathcal{F}(D)$ since $Au = f$ by the consequence of Theorem 6.2.3. Moreover, for $x \in D^{1,1}_{r_k/4}$ we have $u_k^- \in C^2(x)$ since we know that $u_k^- \equiv 0$ in $B(x_0, r_k)$ by the induction hypothesis. Thus, we have $Au_k^-(x) = Lu_k^-(x)$ by (6.2.3), which implies that Au_k^- is well-defined in $D^{1,1}_{r_k/4}$, and so is Au_k^+. We will apply Lemmas Lemma 6.5.2 and Lemma 6.5.3 for the function u_k^+ and $r = r_k/4$ to find m_{k+1} and M_{k+1}. By (6.4.9) and Lemma 6.4.5, we have

$$\begin{aligned}|Au_k^+| \le |Au_k| + |Au_k^-| &\le \left|\frac{1}{C_1}Au - m_k AV(\psi)\right| + |Au_k^-| \\ &\le \left(\frac{1}{C_1}|f| + V(\rho_1/16)|L(V(\psi))|\right) + |Au_k^-| \le C + |Au_k^-|\end{aligned}$$

in D. Thus, we need to estimate $|Au_k^-|$ in $D^{1,1}_{r_k/4}$ in order to use Lemma 6.5.2 and Lemma 6.5.3.

Let $x \in D^{1,1}_{r_k/4}$. By the induction hypothesis, we have $u_k^- \equiv 0$ in $B(x_0, r_k)$, which implies that $u_k^- \in C^2(x)$. Thus, we compute the value $Au_k^-(x)$ using the operator L as follows:

$$0 \le Au_k^-(x) = Lu_k^-(x) \le \int_{x+h \notin B_{r_k}} u_k^-(x+h) \frac{J(1)}{|h|^n \varphi(|h|)} \,\mathrm{d}h. \tag{6.5.6}$$

For any $y \in B_{r_0} \setminus B_{r_k}$, there is $0 \le j < k$ such that $y \in B_{r_j} \setminus B_{r_{j+1}}$. Since $C_1^{-1} u \ge m_j V(\psi)$ and $d_D = \psi$ in B_{r_j}, we have

$$\begin{aligned} u_k(y) &\ge (m_j - m_k) V(\psi(y)) \\ &\ge (m_j - M_j + M_k - m_k) V(d_D(y)) \\ &\ge -(V(r_{j+1}/2)^\gamma - V(r_{k+1}/2)^\gamma) V(r_j). \end{aligned}$$

It follows from $r_{j+1} \le |y - x_0| < r_j \le 8|y - x_0| \le 1$ that

$$\begin{aligned} u_k^-(y) &\le C \left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma \right) V(8|y-x_0|) \\ &\le C \left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma \right) V(|y-x_0|/2). \end{aligned} \tag{6.5.7}$$

Note that (6.5.7) possibly with a larger constant also holds for $y \in \mathbb{R}^n \setminus B_{r_0}$ because $\|u_k\|_{C(\mathbb{R}^n)} \le C$ for all k and

$$\begin{aligned} &\left(V(|y-x_0|/2)^\gamma - V(r_k/16)^\gamma \right) V(|y-x_0|/2) \\ &\ge \left(V(\rho_1/2)^\gamma - V(\rho_1/16)^\gamma \right) V(\rho_1/2) > 0. \end{aligned}$$

Thus, by (6.5.6) and (6.5.7), we have

$$\begin{aligned} &|Au_k^-(x)| \\ &\le C \int_{x+y \notin B_{r_k}} \left(V(|x+h-x_0|/2)^\gamma - V(r_k/16)^\gamma \right) \frac{V(|x+y-x_0|/2)}{|h|^n \varphi(|h|)} \,\mathrm{d}h. \end{aligned}$$

If $x + y \notin B_{r_k}$, then $|h| \ge |x+h-x_0| - |x - x_0| \ge r_k - r_k/2 = r_k/2$ and

$|x+h-x_0| \le r_k/2 + |h| \le 2|h|$. Recalling that $\overline{C}_\varphi(r) = \int_r^\infty \frac{\mathrm{d}s}{s\varphi(s)}$, we obtain

$$\begin{aligned}
|Au_k^-(x)| &\le C \int_{|h|\ge r_k/2} \left(V(|h|)^\gamma - V(r_k/16)^\gamma\right) \frac{V(|h|)}{|h|^n \varphi(|h|)}\,\mathrm{d}h \\
&\le C \int_{r_k/2}^\infty \left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s)\,\mathrm{d}(-\overline{C}_\varphi)(s) \\
&= C\Bigg(\left[-\left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s)\overline{C}_\varphi(s)\right]_{r_k/2}^\infty \\
&\qquad + \int_{r_k/2}^\infty \left((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma\right) V'(s)\overline{C}_\varphi(s)\,\mathrm{d}s\Bigg) \\
&= C(I_1 + I_2).
\end{aligned}$$

By (6.1.9) we have

$$\lim_{s\to\infty} \left(V(s)^\gamma - V(r_k/16)^\gamma\right) V(s)\overline{C}_\varphi(s) \le C \lim_{s\to\infty} \frac{V(s)^\gamma - V(r_k/16)^\gamma}{V(s)} = 0,$$

and hence,

$$I_1 \le C \frac{V(r_k/2)^\gamma - V(r_k/16)^\gamma}{V(r_k/2)}.$$

Using (6.1.9) again, we also have

$$\begin{aligned}
I_2 &\le C \int_{r_k/2}^\infty \left((1+\gamma)V(s)^\gamma - V(r_k/16)^\gamma\right) \frac{V'(s)}{V(s)^2}\,\mathrm{d}s \\
&= C\left(\frac{1+\gamma}{1-\gamma} V(r_k/2)^\gamma - V(r_k/16)^\gamma\right) \frac{1}{V(r_k/2)}.
\end{aligned}$$

Therefore, combining above two inequalities and using (6.1.4) we obtain

$$\begin{aligned}
|Au_k^-(x)| &\le C\left(\frac{2}{1-\gamma} V(r_k/2)^\gamma - 2V(r_k/16)^\gamma\right) \frac{1}{V(r_k/2)} \\
&\le C\left(\frac{2}{1-\gamma}\left(a_3 64^{\sigma_2}\right)^\gamma - 2\left(a_3^{-1} 8^{\sigma_1}\right)^\gamma\right) \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)} \\
&=: C\varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)},
\end{aligned}$$

where $\varepsilon_\gamma \to 0$ as $\gamma \to 0$. We have shown that

$$\|Au_k^+\|_{L^\infty(D^{1,1}_{r_k/4})} \le C\left(1 + \varepsilon_\gamma \frac{V(r_{k+2}/2)^\gamma}{V(r_k/4)}\right).$$

We apply Lemma 6.5.2 and Lemma 6.5.3 to $u_k^+ \in \mathcal{F}(D, D^{1,1}_{r_k/4})$. Since $u_k = u_k^+$ and $d_D = \psi$ in D_{r_k}, we have

$$\begin{aligned}
&\sup_{D^+_{\kappa' r_k/4}} \left(\frac{u}{C_1 V(\psi)} - m_k\right) \\
&\le C\left(\inf_{D^+_{\kappa' r_k/4}} \left(\frac{u}{C_1 V(\psi)} - m_k\right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma\right) \\
&\le C\left(\inf_{D_{r_{k+1}}} \left(\frac{u}{C_1 V(\psi)} - m_k\right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma\right).
\end{aligned}$$

Repeating this procedure with the function $u_k = M_k V(d_D) - C_1^{-1} u$ instead of $u_k = C_1^{-1} u - m_k V(d_D)$, we also have

$$\begin{aligned}
&\sup_{D^+_{\kappa' r_k/4}} \left(M_k - \frac{u}{C_1 V(\psi)}\right) \\
&\le C\left(\inf_{D_{r_{k+1}}} \left(M_k - \frac{u}{C_1 V(\psi)}\right) + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma\right).
\end{aligned}$$

Adding up these two inequalities, we obtain

$$\begin{aligned}
M_k - m_k \le C\Bigg(&\inf_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} - \sup_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} \\
&+ M_k - m_k + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma\Bigg).
\end{aligned}$$

Thus, recalling that $M_k - m_k = V(r_{k+1}/2)^\gamma$, we get

$$\begin{aligned}&\sup_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} - \inf_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} \\ &\leq \frac{C-1}{C} V(r_{k+1}/2)^\gamma + V(r_k/4) + \varepsilon_\gamma V(r_{k+2}/2)^\gamma \\ &\leq \left(\frac{C-1}{C} a_3 8^{\sigma_2 \gamma} + a_3 32^{\sigma_2\gamma} V(\rho_1)^{1-\gamma} + \varepsilon_\gamma \right) V(r_{k+2}/2)^\gamma.\end{aligned}$$

By taking γ and ρ_1 sufficiently small, we have

$$\frac{C-1}{C} a_3 8^{\sigma_2\gamma} + a_3 32^{\sigma_2\gamma} V(\rho_1)^{1-\gamma} + \varepsilon_\gamma \leq 1,$$

and it yields that

$$\sup_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} - \inf_{D_{r_{k+1}}} \frac{u}{C_1 V(\psi)} \leq V(r_{k+2}/2)^\gamma.$$

Therefore, we are able to choose m_{k+1} and M_{k+1}. $\square$

We finally prove the Theorem 6.0.2 using the Lemma 6.5.1.

Proof of Theorem 6.0.2. Let us assume that $\|f\|_{L^\infty(D)} \leq 1$ and $\|u\|_{C(D)} \leq C$ as in the proof of Lemma 6.5.1. We first show that the following holds for any $x \in D$:

$$\left[\frac{u}{V(d_D)} \right]_{C^\beta(\overline{B_{r/2}(x)})} \leq \frac{C}{r^\beta V(r)}$$

for each $0 < \beta \leq \sigma_1$, where $r = d_D(x)$. We are going to use the inequality

$$\left[\frac{u}{V(d_D)} \right]_{C^\beta} \leq \|u\|_C \left[\frac{1}{V(d_D)} \right]_{C^\beta} + [u]_{C^\beta} \left\| \frac{1}{V(d_D)} \right\|_C. \tag{6.5.8}$$

From (6.3.17) we know that $[u]_{C^V(\overline{B_{r/2}(x)})} \leq C$. Thus, we have $[u]_{C^\beta(\overline{B_{r/2}(x)})} \leq C$ for each $0 < \beta \leq \sigma_1$. Since $d_D(y) \geq r/2$ for $y \in B(x, r/2)$, we have

$$\left\| \frac{1}{V(d_D)} \right\|_{C(\overline{B_{r/2}(x)})} \leq \frac{C}{V(r)}$$

and

$$\begin{aligned}
\left[\frac{1}{V(d_D)}\right]_{C^{0,1}(\overline{B_{r/2}(x)})} &\leq \sup_{y,z\in B_{r/2}(x)} \frac{|V(d_D(y))^{-1}-V(d_D(z))^{-1}|}{|y-z|}\\
&\leq \sup_{y,z\in B_{r/2}(x)} \frac{V'(d^*)}{V(d^*)^2}\frac{|d_D(y)-d_D(z)|}{|y-z|}\\
&\leq C\left(\sup_{y,z\in B(x,r/2)} \frac{1}{d^*V(d^*)}\right)[d]_{C^{0,1}(\overline{B_{r/2}(x)})} \leq \frac{C}{rV(r)},
\end{aligned}$$

where d^* is a value in between $d_D(y)$ and $d_D(z)$], so $d^* \geq r/2$. Thus, by the interpolation, we obtain

$$\left[\frac{1}{V(d_D)}\right]_{C^{\beta}(\overline{B_{r/2}(x)})} \leq C\left\|\frac{1}{V(d_D)}\right\|^{1-\beta}_{C(\overline{B_{r/2}(x)})}\left[\frac{1}{V(d_D)}\right]^{\beta}_{C^{0,1}(\overline{B_{r/2}(x)})} \leq \frac{C}{r^{\beta}V(r)}$$

and it follows from (6.5.8) that

$$\left[\frac{u}{V(d_D)}\right]_{C^{\beta}} \leq \frac{C}{r^{\beta}V(r)}+\frac{C}{V(r)} \leq \frac{C}{r^{\beta}V(r)}. \tag{6.5.9}$$

Let $x, y \in D$ and let us next show that

$$\left|\frac{u(x)}{V(d_D(x))}-\frac{u(y)}{V(d_D(y))}\right| \leq C|x-y|^{\alpha}$$

for some $\alpha > 0$. Without loss of generality, we may assume that $r := d_D(x) \geq d_D(y)$. Fix any $0 < \beta \leq \sigma_1$ and let $p > 1 + \sigma_2/\beta$. If $|x-y| \leq r^p/2$, then we have $|x-y| \leq r/2$ and $y \in B(x, r/2)$ since $r \leq 1$. Thus, by (6.5.9) we obtain

$$\left|\frac{u(x)}{V(d_D(x))}-\frac{u(y)}{V(d_D(y))}\right| \leq C\frac{|x-y|^{\beta}}{r^{\beta}V(r)} \leq C\frac{|x-y|^{\beta-\beta/p}}{V(|x-y|^{1/p})} \leq C|x-y|^{\beta-\frac{\beta+\sigma_2}{p}}.$$

On the other hand, if $|x-y| \geq r^p/2$, let $x_0, y_0 \in \partial D$ be boundary points satisfying $d_D(x) = |x-x_0|$ and $d_D(y) = |y-y_0|$. Then by Lemma 6.5.1 we have

$$\left|\frac{u(x)}{V(d_D)(x)}-\frac{u(x_0)}{V(d_D)(x_0)}\right| \leq CV(d_D(x))^{\gamma}, \tag{6.5.10}$$

$$\left|\frac{u(y)}{V(d_D)(y)} - \frac{u(y_0)}{V(d_D)(y_0)}\right| \le CV(d_D(y))^\gamma, \tag{6.5.11}$$

and

$$\left|\frac{u(x_0)}{V(d_D)(x_0))} - \frac{u(y_0)}{V(d_D)(y_0)}\right| \le CV(d_D(x) + |x-y| + d_D(y))^\gamma. \tag{6.5.12}$$

Using inequalities (6.5.10), (6.5.11), and (6.5.12) we obtain

$$\left|\frac{u(x)}{V(d_D)(x))} - \frac{u(y)}{V(d_D)(y)}\right| \le C\left(2V(r)^\gamma + V(2r + |x-y|)^\gamma\right) \le C|x-y|^{\sigma_1\gamma/p}.$$

Therefore, taking $\alpha = \min\{\beta - (\beta+\sigma_2)/p, \sigma_1\gamma/p\}$ finishes the result. $\square$

Appendix A

Asymptotics of the Constant $C(n, \varphi)$

It is well-known that the constant $C(n, \sigma)$ for the fractional Laplacian has the asymptotic properties

$$\lim_{\sigma \to 2^-} \frac{C(n,\sigma)}{2-\sigma} = \frac{2}{\omega_n} \quad \text{and} \quad \lim_{\sigma \to 0^+} \frac{C(n,\sigma)}{\sigma} = \frac{1}{n\omega_n}, \tag{A.0.1}$$

where ω_n denotes the volume of the n-dimensional unit ball. Moreover, it is also known that for each $x \in \mathbb{R}^n$ and a function u regular enough,

$$\lim_{\sigma \to 2^-} (-\Delta)^{\sigma/2} u(x) = -\Delta u(x) \quad \text{and} \quad \lim_{\sigma \to 0^+} (-\Delta)^{\sigma/2} u(x) = u(x). \tag{A.0.2}$$

See [31] for the proofs of (A.0.1) and (A.0.2). In this section, we state and prove analogues of (A.0.1) and (A.0.2), which will imply that the constant $C(n, \varphi)$ generalizes $C(n, \sigma)$ under the weak scaling condition (2.1.1).

In order to state analogues of (A.0.1) and (A.0.2), we need to clarify the meaning of the limits

$$\varphi(r) \to r^2 \quad \text{and} \quad \varphi(r) \to r^0. \tag{A.0.3}$$

This can be done by considering a sequence of functions φ_k satisfying the weak scaling conditions (2.1.1) with sequences of constants $0 < \underline{\sigma}_k \leq \overline{\sigma}_k < 2$ and $a_k \geq 1$. That is, limits in (A.0.3) can be understood as limits

$$\underline{\sigma}_k, \overline{\sigma}_k \to 2 \;\&\; a_k \to 1 \quad \text{and} \quad \underline{\sigma}_k, \overline{\sigma}_k \to 0 \;\&\; a_k \to 1,$$

respectively. Let us consider a sequence of operators

$$L_k u(x) = \frac{1}{2} C(n, \varphi_k) \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{|y|^n \varphi_k(|y|)} \, \mathrm{d}y,$$

and assume that

$$\lim_{k \to \infty} a_k = 1 \tag{A.0.4}$$

throughout this section. The following propositions correspond to (A.0.1) and (A.0.2). Recall that $\underline{C}(R) = \frac{R^{2-\sigma}}{2-\sigma}$ and $\overline{C}(R) = \frac{R^{-\sigma}}{\sigma}$ in the case of the fractional Laplacian.

Proposition A.0.1. *Assume that* (A.0.4) *holds. If* $\lim_{k \to \infty} \underline{\sigma}_k = \lim_{k \to \infty} \overline{\sigma}_k = 2$*, then*

$$\lim_{k \to \infty} C(n, \varphi_k) \underline{C}_{\varphi_k}(R) = \frac{2}{\omega_n}, \tag{A.0.5}$$

and if $\lim_{k \to \infty} \underline{\sigma}_k = \lim_{k \to \infty} \overline{\sigma}_k = 0$*, then*

$$\lim_{k \to \infty} C(n, \varphi_k) \overline{C}_{\varphi_k}(R) = \frac{1}{n \omega_n}. \tag{A.0.6}$$

Proposition A.0.2. *Assume that* (A.0.4) *holds. Let* $u \in C_c^\infty(\mathbb{R}^n)$ *and* $x \in \mathbb{R}^n$*. If* $\lim_{k \to \infty} \underline{\sigma}_k = \lim_{k \to \infty} \overline{\sigma}_k = 2$*, then*

$$\lim_{k \to \infty} -L_k u(x) = -\Delta u(x),$$

and if $\lim_{k \to \infty} \underline{\sigma}_k = \lim_{k \to \infty} \overline{\sigma}_k = 0$*, then*

$$\lim_{k \to \infty} -L_k u(x) = u(x).$$

Proof of Proposition A.0.1. Let us first assume that

$$\lim_{k\to\infty} \underline{\sigma}_k = \lim_{k\to\infty} \overline{\sigma}_k = 2. \tag{A.0.7}$$

By writing $C(n,\varphi_k)^{-1}$ as a double integral, we have

$$C(n,\varphi_k)^{-1} = \int_{\mathbb{R}^{n-1}} \frac{1}{\zeta^n} \int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r\mathrm{d}y',$$

where $\zeta = \zeta(y') = (1+|y'|^2)^{1/2}$. We use the inequality (2.1.4) to estimate

$$0 \le \int_{|r|\ge R} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r \le \int_R^\infty \frac{4}{r\varphi_k(r)}\,\mathrm{d}r \le \frac{4a_k}{\underline{\sigma}_k\varphi_k(R)}.$$

Using the assumptions (A.0.4), (A.0.7), and the inequality (2.1.3), we obtain

$$0 \le \lim_{k\to\infty} \frac{1}{\underline{C}_{\varphi_k}(R)} \int_{|r|\ge R} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r \le \lim_{k\to\infty} 4a_k^2 \frac{2-\underline{\sigma}_k}{\underline{\sigma}_k} R^{-2} = 0.$$

On the other hand, using the weak scaling condition (2.1.1), we have

$$\begin{aligned}\left|\int_{|r|<R} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r - \int_{|r|<R} \frac{r^2}{2\zeta^2|r|\varphi_k(|r|)}\,\mathrm{d}r\right| &\le \frac{1}{24}\int_{|r|<R} \frac{r^4}{\zeta^4|r|\varphi_k(|r|)}\,\mathrm{d}r \\ &\le \frac{a_k}{12\zeta^4(4-\overline{\sigma}_k)} \frac{R^4}{\varphi_k(R)}.\end{aligned}$$

Since

$$0 \le \lim_{k\to\infty} \frac{1}{\underline{C}_{\varphi_k}(R)} \frac{a_k}{12\zeta^4(4-\overline{\sigma}_k)} \frac{R^4}{\varphi_k(R)} \le \lim_{k\to\infty} \frac{a_k^2}{12\zeta^2} \frac{2-\underline{\sigma}_k}{4-\underline{\sigma}_k} R^2 = 0,$$

we obtain

$$\lim_{k\to\infty} \frac{1}{\underline{C}_{\varphi_k}(R)} \int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r = \lim_{k\to\infty} \frac{1}{\underline{C}_{\varphi_k}(R)} \frac{1}{\zeta^2} \int_0^R \frac{r}{\varphi_k(r)}\,\mathrm{d}r = \frac{1}{\zeta^2}.$$

Therefore, we conclude that

$$\begin{aligned}\lim_{k\to\infty} C(n,\varphi_k)\underline{C}_{\varphi_k}(R) &= \lim_{k\to\infty}\left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}}\frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r\frac{\mathrm{d}y'}{\zeta^n}\right)^{-1}\underline{C}_{\varphi_k}(R)\\ &= \left(\int_{\mathbb{R}^{n-1}}\lim_{k\to\infty}\frac{1}{\underline{C}_{\varphi_k}(R)}\int_{\mathbb{R}}\frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r\frac{\mathrm{d}y'}{\zeta^n}\right)^{-1}\\ &= \left(\int_{\mathbb{R}^{n-1}}\frac{1}{\zeta^{n+2}}\,\mathrm{d}y'\right)^{-1} = \frac{2}{\omega_n}.\end{aligned}$$

See [31, Corollary 4.2] for the last equality.

Let us next assume that

$$\lim_{k\to\infty}\underline{\sigma}_k = \lim_{k\to\infty}\overline{\sigma}_k = 0. \tag{A.0.8}$$

We use the inequality (2.1.3) to estimate

$$0 \le \int_{|r|<R}\frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r \le \frac{1}{\zeta^2}\int_0^R\frac{r}{\varphi_k(r)}\,\mathrm{d}r \le \frac{a_k}{\zeta^2(2-\overline{\sigma}_k)}\frac{R^2}{\varphi_k(R)}.$$

Using the assumptions (A.0.4), (A.0.8), and the inequality (2.1.4), we obtain

$$\lim_{k\to\infty}\frac{1}{\overline{C}_{\varphi_k}(R)}\int_{|r|<R}\frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r \le \lim_{k\to\infty}\frac{a_k^2}{\zeta^2}\frac{\overline{\sigma}_k}{2-\overline{\sigma}_k}R^2 = 0. \tag{A.0.9}$$

On the other hand, observe that for any integer $m\ge 1$, we have

$$\begin{aligned}\left|\int_{2m\zeta\pi}^{2(m+1)\zeta\pi}\frac{\cos(r/\zeta)}{r\varphi_k(r)}\,\mathrm{d}r\right| &= \left|\int_{2m\zeta\pi}^{(2m+1)\zeta\pi}\left(\frac{\cos(r/\zeta)}{r\varphi_k(r)}+\frac{\cos((r/\zeta)+\pi)}{(r+\zeta\pi)\varphi_k(r+\zeta\pi)}\right)\mathrm{d}r\right|\\ &\le \int_{2m\zeta\pi}^{(2m+1)\zeta\pi}\left|\frac{1}{r\varphi_k(r)}-\frac{1}{(r+\zeta\pi)\varphi_k(r+\zeta\pi)}\right|\mathrm{d}r\\ &= \int_{2m}^{2m+1}\left|\frac{1}{r\varphi_k(\zeta\pi r)}-\frac{1}{(r+1)\varphi_k(\zeta\pi(r+1))}\right|\mathrm{d}r.\end{aligned} \tag{A.0.10}$$

Let us write $A=\frac{1}{r\varphi_k(\zeta\pi r)}-\frac{1}{(r+1)\varphi_k(\zeta\pi(r+1))}$. If $A\ge 0$, then by the weak scaling condition (2.1.1), we have for $r \in [2m, 2m+1]$

$$\begin{aligned}
|A| &\leq \frac{a_k}{\varphi_k((2m+1)\zeta\pi)}\frac{1}{r}\left(\frac{2m+1}{r}\right)^{\overline{\sigma}_k} - \frac{a_k^{-1}}{\varphi_k((2m+1)\zeta\pi)}\frac{1}{r+1}\left(\frac{2m+1}{r+1}\right)^{\overline{\sigma}_k} \\
&= \frac{(2m+1)^{\overline{\sigma}_k}}{a_k\varphi_k((2m+1)\zeta\pi)}\frac{a_k^2(r+1)^{1+\overline{\sigma}_k} - r^{1+\overline{\sigma}_k}}{r^{1+\overline{\sigma}_k}(r+1)^{1+\overline{\sigma}_k}} \\
&\leq \frac{1}{2a_k m\varphi((2m+1)\zeta\pi)}\left(\frac{2m+1}{2m}\right)^{\overline{\sigma}_k}\frac{a_k^2(r+1)^{1+\overline{\sigma}_k} - r^{1+\overline{\sigma}_k}}{(r+1)^{1+\overline{\sigma}_k}} \\
&\leq \frac{9}{8a_k m\varphi((2m+1)\zeta\pi)}\frac{a_k^2(r+1)^{1+\overline{\sigma}_k} - r^{1+\overline{\sigma}_k}}{(r+1)^{1+\overline{\sigma}_k}}.
\end{aligned}$$

If $A \leq 0$, then we have

$$|A| \leq \frac{9}{8a_k m\varphi_k((2m+1)\zeta\pi)}\frac{a_k^2 r^{1+\underline{\sigma}_k} - (r+1)^{1+\underline{\sigma}_k}}{(r+1)^{1+\underline{\sigma}_k}}$$

by a similar argument. Since

$$\begin{aligned}
\frac{a_k^2(r+1)^{1+\overline{\sigma}_k} - r^{1+\overline{\sigma}_k}}{(r+1)^{1+\overline{\sigma}_k}} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left((1+(a_k^2-1)s)(r+s)^{1+\overline{\sigma}_k}\right)\frac{\mathrm{d}s}{(r+1)^{1+\overline{\sigma}_k}} \\
&= \int_0^1 \frac{(a_k^2-1)(r+s)^{1+\overline{\sigma}_k}}{(r+1)^{1+\overline{\sigma}_k}}\,\mathrm{d}s \\
&\quad + \int_0^1 \frac{(1+\overline{\sigma}_k)(1+(a_k^2-1)s)(r+s)^{\overline{\sigma}_k}}{(r+1)^{1+\overline{\sigma}_k}}\,\mathrm{d}s \\
&\leq a_k^2 - 1 + \frac{3a_k^2}{r+1}
\end{aligned}$$

and

$$\begin{aligned}
\frac{a_k^2 r^{1+\underline{\sigma}_k} - (r+1)^{1+\underline{\sigma}_k}}{(r+1)^{1+\underline{\sigma}_k}} &= \int_0^1 \frac{\mathrm{d}}{\mathrm{d}s}\left((a_k^2 s + 1 - s)(r+1-s)^{1+\underline{\sigma}_k}\right)\frac{\mathrm{d}s}{(r+1)^{1+\underline{\sigma}_k}} \\
&= \int_0^1 \frac{(a_k^2-1)(r+1-s)^{1+\underline{\sigma}_k}}{(r+1)^{1+\underline{\sigma}_k}}\,\mathrm{d}s \\
&\quad - \int_0^1 \frac{(1+\underline{\sigma}_k)(a_k^2 s+1-s)(r+1-s)^{\underline{\sigma}_k}}{(r+1)^{1+\underline{\sigma}_k}}\,\mathrm{d}s \\
&\leq a_k^2 - 1,
\end{aligned}$$

we further estimate the integrand $|A|$ in the right-hand side of (A.0.10) as

$$|A| \leq \frac{9}{8a_k m\varphi((2m+1)\zeta\pi)} \left(a_k^2 - 1 + \frac{3a_k^2}{r+1}\right) \tag{A.0.11}$$

regardless of the sign of A.

Let $N \geq 1$ be the integer satisfying $2(N-1)\zeta\pi < R \leq 2N\zeta\pi$. Then, it follows from (A.0.10) and (A.0.11) that, we have for $m \geq N$

$$\begin{aligned}\left|\int_{2m\zeta\pi}^{2(m+1)\zeta\pi} \frac{\cos(r/\zeta)}{r\varphi_k(r)}\,\mathrm{d}r\right| &\leq \frac{9}{8a_k m\varphi_k((2m+1)\zeta\pi)} \left(a_k^2 - 1 + \frac{3a_k^2}{2m+1}\right) \\ &\leq \frac{9(a_k^2-1)}{8a_k} \frac{1}{m\varphi_k((2m+1)\zeta\pi)} + \frac{27a_k^2}{16\varphi_k(R)} \frac{1}{m^2},\end{aligned}$$

and hence,

$$\sum_{m=N}^{\infty} \left|\int_{2m\zeta\pi}^{2(m+1)\zeta\pi} \frac{\cos(r/\zeta)}{r\varphi_k(r)}\,\mathrm{d}r\right| \leq \sum_{m=N}^{\infty} \frac{9(a_k^2-1)/(8a_k)}{m\varphi_k((2m+1)\zeta\pi)} + \frac{9\pi^2 a_k^2}{32\varphi_k(R)}.$$

Since

$$\int_R^{2N\zeta\pi} \frac{\mathrm{d}r}{r\varphi_k(r)} \leq \frac{a_k}{\varphi_k(R)} \int_R^{2N\zeta\pi} \frac{\mathrm{d}r}{r} \leq \frac{a_k}{\varphi_k(R)} \log\frac{2N\zeta\pi}{R},$$

we have obtained that

$$\left|\int_R^{\infty} \frac{\cos(r/\zeta)}{r\varphi_k(r)}\,\mathrm{d}r\right| \leq \frac{a_k}{\varphi_k(R)} \log\frac{2N\zeta\pi}{R} + \sum_{m=N}^{\infty} \frac{9(a_k^2-1)/(8a_k)}{m\varphi_k((2m+1)\zeta\pi)} + \frac{9\pi^2 a_k^2}{32\varphi_k(R)}. \tag{A.0.12}$$

We claim that

$$\lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \int_R^{\infty} \frac{\cos(r/\zeta)}{r\varphi_k(r)}\,\mathrm{d}r = 0. \tag{A.0.13}$$

Indeed, by using the assumptions (A.0.4), (A.0.8), and the inequality (2.1.4), the first and the third terms in the right-hand side of (A.0.12) can be handled as

$$0 \leq \lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \frac{a_k}{\varphi_k(R)} \log\frac{2N\zeta\pi}{R} \leq \lim_{k\to\infty} a_k^2 \overline{\sigma}_k \log\frac{2N\zeta\pi}{R} = 0$$

and

$$0 \le \lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \frac{9\pi^2 a_k^2}{32\varphi_k(R)} \le \lim_{k\to\infty} \frac{9\pi^2 a_k^3}{32}\overline{\sigma}_k = 0.$$

For the second term, we first observe that

$$\begin{aligned}
\overline{C}_{\varphi_k}(R) &= \int_R^\infty \frac{1}{r\varphi_k(r)}\,\mathrm{d}r \ge \sum_{m=N}^\infty \int_{2m\zeta\pi}^{2(m+1)\zeta\pi} \frac{1}{r\varphi_k(r)}\,\mathrm{d}r \\
&\ge \frac{1}{a_k}\sum_{m=N}^\infty \int_{2m\zeta\pi}^{2(m+1)\zeta\pi} \frac{1}{2(m+1)\zeta\pi\varphi_k(2(m+1)\zeta\pi)}\,\mathrm{d}r \\
&= \frac{1}{a_k}\sum_{m=N}^\infty \frac{1}{(m+1)\varphi_k(2(m+1)\zeta\pi)}.
\end{aligned}$$

Since $m+1 \le 2m$ and

$$\frac{\varphi_k(2(m+1)\zeta\pi)}{\varphi_k((2m+1)\zeta\pi)} \le a_k\left(\frac{2m+2}{2m+1}\right)^{\overline{\sigma}_k} \le a_k\left(\frac{4}{3}\right)^2,$$

we have

$$\overline{C}_{\varphi_k}(R) \ge \frac{9}{16a_k^2}\sum_{m=N}^\infty \frac{1}{m\varphi_k((2m+1)\zeta\pi)},$$

which yields that

$$\frac{1}{\overline{C}_{\varphi_k}(R)}\sum_{m=N}^\infty \frac{1}{m\varphi_k((2m+1)\zeta\pi)} \le \frac{16a_k^2}{9}.$$

Thus, we obtain

$$0 \le \lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \frac{9(a_k^2-1)}{8a_k}\sum_{m=N}^\infty \frac{1}{m\varphi_k((2m+1)\zeta\pi)} \le \lim_{k\to\infty} 2a_k(a_k^2-1) = 0,$$

which proves the claim. Combining (A.0.9) and (A.0.13), we arrive at

$$\lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r = \lim_{k\to\infty} \frac{1}{\overline{C}_{\varphi_k}(R)} \int_{|r|\ge R} \frac{dr}{|r|\varphi_k(|r|)} = 2.$$

Therefore, we conclude that

$$\begin{aligned}
\lim_{k\to\infty} C(n,\varphi_k)\overline{C}_{\varphi_k}(R) &= \lim_{k\to\infty}\left(\int_{\mathbb{R}^{n-1}}\int_{\mathbb{R}} \frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r\frac{\mathrm{d}y'}{\zeta^n}\right)^{-1}\overline{C}_{\varphi_k}(R)\\
&= \left(\int_{\mathbb{R}^{n-1}}\lim_{k\to\infty}\frac{1}{\overline{C}_{\varphi_k}(R)}\int_{\mathbb{R}}\frac{1-\cos(r/\zeta)}{|r|\varphi_k(|r|)}\,\mathrm{d}r\frac{\mathrm{d}y'}{\zeta^n}\right)^{-1}\\
&= \left(\int_{\mathbb{R}^{n-1}}\frac{2}{\zeta^n}\,\mathrm{d}y'\right)^{-1} = \frac{1}{n\omega_n},
\end{aligned}$$

which finishes the proof. See [31, Corollary 4.2] for the last equality. □

We close this chapter with the proof of Proposition A.0.2.

Proof of Proposition A.0.2. We assume first that $\lim_{k\to\infty}\underline{\sigma}_k = \lim_{k\to\infty}\overline{\sigma}_k = 2$. In this case, we have no contribution outside the unit ball. Indeed, using the inequality (2.1.4) we have

$$\left|-\int_{B_1^c}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| \le 4n\omega_n\|u\|_{L^\infty(\mathbb{R}^n)}\int_1^\infty\frac{\mathrm{d}r}{r\varphi_k(r)} \le \frac{4n\omega_n a_k}{\underline{\sigma}_k\varphi_k(1)}\|u\|_{L^\infty(\mathbb{R}^n)}.$$

Hence, using the inequality (2.1.3) and the limit (A.0.5) we obtain

$$\begin{aligned}
\left|-\frac{1}{2}C_{\varphi_k}\int_{B_1^c}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| &= \frac{1}{2\underline{C}_{\varphi_k}(1)}C(n,\varphi_k)\underline{C}_{\varphi_k}(1)\left|\int_{B_1^c}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right|\\
&\le 2n\omega_n a_k^2\frac{2-\underline{\sigma}_k}{\underline{\sigma}_k}C(n,\varphi_k)\underline{C}_{\varphi_k}(1)\|u\|_{L^\infty(\mathbb{R}^n)} \to 0
\end{aligned}$$

as $k\to\infty$. On the other hand, we have

$$\begin{aligned}
\left|\int_{B_1}\frac{\delta(u,x,y)-y\cdot D^2u(x)y}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| &\le \|u\|_{C^3(\overline{B_1})}\int_{B_1}\frac{|y|^3}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\\
&\le n\omega_n\|u\|_{C^3(\overline{B_1})}\int_0^1\frac{r^2}{\varphi_k(r)}\,\mathrm{d}r\\
&\le \frac{n\omega_n a_k}{\varphi_k(1)(3-\overline{\sigma}_k)}\|u\|_{C^3(\overline{B_1})},
\end{aligned}$$

and this implies that

$$\lim_{k\to\infty} -\frac{1}{2}C_{\varphi_k}\int_{B_1}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y = \lim_{k\to\infty} -\frac{1}{2}C_{\varphi_k}\int_{B_1}\frac{y\cdot D^2u(x)y}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y.$$

Note that if $i \neq j$ then

$$\int_{B_1}\frac{D_{ij}u(x)y_iy_j}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y = -\int_{B_1}\frac{D_{ij}u(x)\tilde{y}_i\tilde{y}_j}{|\tilde{y}|^n\varphi_k(|\tilde{y}|)}\,\mathrm{d}\tilde{y},$$

where $\tilde{y}_j = -y_j$ and $\tilde{y}_k = \tilde{y}_k$ for any $k \neq j$. This implies that

$$\int_{B_1}\frac{D_{ij}u(x)y_iy_j}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y = 0,$$

and hence,

$$\begin{aligned}\int_{B_1}\frac{y\cdot D^2u(x)y}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y &= \sum_{i=1}^n D_{ii}u(x)\int_{B_1}\frac{y_i^2}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\\ &= \sum_{i=1}^n \frac{D_{ii}u(x)}{n}\int_{B_1}\frac{|y|^2}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\\ &= \omega_n\Delta u(x)\int_0^1\frac{r}{\varphi_k(r)}\,\mathrm{d}r.\end{aligned}$$

Using (A.0.5), we conclude that

$$\lim_{k\to\infty} -L_ku(x) = \lim_{k\to\infty}\frac{\omega_n}{2}C(n,\varphi_k)\underline{C}_{\varphi_k}(1)(-\Delta u)(x) = -\Delta u(x).$$

Let us next assume that $\lim\limits_{k\to\infty}\underline{\sigma}_k = \lim\limits_{k\to\infty}\overline{\sigma}_k = 0$. Fix $x \in \mathbb{R}^n$ and let $R_0 > 0$ be such that $\operatorname{supp} u \subset B_{R_0}$ and set $R = R_0 + |x| + 1$. Then, by using the inequality (2.1.3), we have

$$\left|\int_{B_R}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| \leq n\omega_n\|u\|_{C^2(\overline{B_R})}\int_0^R\frac{r}{\varphi_k(r)}\,\mathrm{d}r \leq \frac{n\omega_na_k}{2-\overline{\sigma}_k}\frac{R^2}{\varphi_k(R)}\|u\|_{C^2(\overline{B_R})}.$$

Hence, using the inequality (2.1.4) and the limit (A.0.6), we obtain

$$\begin{aligned}\left|-\frac{C(n,\varphi_k)}{2}\int_{B_R}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| &\le \frac{C(n,\varphi_k)\overline{C}_{\varphi_k}(R)}{2\overline{C}_{\varphi_k}(R)}\left|\int_{B_R}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y\right| \\ &\le \frac{n\omega_n a_k^2R^2\overline{\sigma}_k}{2(2-\overline{\sigma}_k)}C(n,\varphi_k)\overline{C}_{\varphi_k}(R)\|u\|_{C^2(\mathbb{R}^n)}\to 0\end{aligned}$$

as $k\to\infty$. On the other hand, if $|y|\ge R$, then we have $|x+y|>R_0$ and $|x-y|>R$, and consequently $u(x+y)=u(x-y)=0$. Thus, we have

$$-\frac{1}{2}C_{\varphi_k}\int_{\mathbb{R}^n\setminus B_R}\frac{\delta(u,x,y)}{|y|^n\varphi_k(|y|)}\,\mathrm{d}y = n\omega_nC(n,\varphi_k)\overline{C}_{\varphi_k}(R)u(x).$$

Therefore, using (A.0.6) we conclude that

$$\lim_{k\to\infty}-L_ku(x)=n\omega_n\lim_{k\to\infty}C(n,\varphi_k)\overline{C}_{\varphi_k}(R)u(x)=u(x),$$

finishing the proof. $\square$

Appendix B

Inequalities

We record some inequalities and the fractional Sobolev inequalities that are be used in Chapter 5. Let us first provide some algebraic inequalities.

Lemma B.0.1 (Lemma 3.2 in [47]). *For $a, b \geq 0$ and $s \in (0,1]$*

$$\frac{8}{2^{1/s}}(1-s)^{-2}\left((1+b)^{(1-s)/2}-(1+a)^{(1-s)/2}\right)^2 \\ \leq (b-a)\left(b(1+b^s)^{-1/s}-a(1+a^s)^{-1/s}\right).$$

Lemma B.0.2. *For $a, b > 0$*

$$(b-a)(a^{-1}-b^{-1}) \geq (\log b - \log a)^2.$$

One can easily prove Lemma B.0.2 by setting $a = e^x$, $b = e^y$, and then applying the Taylor expansion.

Lemma B.0.3 (Lemma 3.7 in [45]). *Let $a, b \geq 0, \eta_1, \eta_2 \geq 0$, and let $q > 1$. Then*

$$(b-a)(\eta_2^2 b^{q-1} - \eta_1^2 a^{q-1}) \geq \frac{q-1}{32q^2}\left(\eta_2 b^{\frac{q}{2}} - \eta_1 a^{\frac{q}{2}}\right)^2 \\ - 2(1 \vee (q-1)^{-1})(\eta_2-\eta_1)^2(b^q+a^q).$$

Lemma B.0.4. *For $a, b, x, y \in \mathbb{R}$*

$$(b-a)(by^2 - ax^2) \geq \frac{1}{4}(b-a)^2(y^2+x^2) - 4(b^2+a^2)(y-x)^2.$$

Proof. Since $by^2 - ax^2 = \frac{1}{2}(b-a)(y^2+x^2) + \frac{1}{2}(b+a)(y^2-x^2)$, we have

$$\begin{aligned}(b-a)(by^2 - ax^2) &= \frac{1}{2}(b-a)^2(y^2+x^2) + \frac{1}{2}(b^2-a^2)(y^2-x^2) \\ &\geq \frac{1}{4}(b-a)^2(y^2+x^2) - 4(b^2+a^2)(y-x)^2.\end{aligned}$$

□

The next lemma shows the relation between L^p spaces and L^p_{weak} spaces, and their norms. See, for example, [56, Theorem 2.18.8] for the proof of Lemma B.0.5.

Lemma B.0.5. *Let Ω be a bounded open set in $\mathbb{R}^n$ and let $p \geq 1$. Then*

$$\begin{aligned}&L^p(\Omega) \subset L^p_{\text{weak}}(\Omega) \quad \textit{with} \quad [f]_{L^p_{\text{weak}}(\Omega)} \leq \|f\|_{L^p(\Omega)}, \quad \textit{and} \\ &L^p_{\text{weak}}(\Omega) \subset L^q(\Omega) \quad \textit{with} \quad \|f\|_{L^q(\Omega)} \leq \left(\frac{p}{p-q}\right)^{\frac{1}{q}} |\Omega|^{\frac{1}{q}-\frac{1}{p}} [f]_{L^p_{\text{weak}}(\Omega)},\end{aligned} \tag{B.0.1}$$

where $1 \leq q < p$.

We close this chapter with the fractional Sobolev inequalities, whose proofs can be found in [27, 13, 61].

Proposition B.0.6 (Fractional Sobolev inequality)**.** *Let $\sigma \in (0,2)$ and $2^\star = 2n/(n-\sigma)$, and assume $n > \sigma$.*

(i) For any measurable and compactly supported function $f : \mathbb{R}^n \to \mathbb{R}$, we have

$$\|f\|^2_{L^{2^\star}(\mathbb{R}^n)} \leq C(2-\sigma) \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(y)-f(x)|^2}{|y-x|^{n+\sigma}} \, \mathrm{d}y \, \mathrm{d}x,$$

for some constant C depending only on n.

(ii) For any measurable function $f : B_r \to \mathbb{R}$*, we have*

$$\|f\|^2_{L^{2^\star}(B_r)} \leq C(2-\sigma)\left(\int_{B_r}\int_{B_r}\frac{(f(y)-f(x))^2}{|y-x|^{n+\sigma}}\,\mathrm{d}y\,\mathrm{d}x + r^{-\sigma}\|f\|^2_{L^2(B_r)}\right),$$

for some constant C *depending only on* n*.*

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국문초록

비국소작용소는 해석학과 확률론 등에서 아주 중요하다. 본 학위논문은 비국소작용소에 대한 내부 및 경계에서의 정칙성을 다룬 네 개의 연구논문으로 구성된다. 함수적 차수의 커널을 갖는 비선형 비국소작용소에 대하여 첫 번째 논문에서는 Krylov–Safonov 이론을, 두 번째 논문에서는 Evans–Krylov 이론과 Schauder 이론을 다룬다. Aleksandrov–Bakelman–Pucci 추정, Harnack 부등식, Hölder 연속성, 일반화된 Hölder 연속성 등의 내부 정칙성을 연구한다. 세 번째 논문에서는 비국소작용소의 그린 함수의 상계와 하계를 순수 해석적 방법을 이용하여 구한다. 위의 세 논문의 핵심은 기존에 잘 알려진 국소작용소의 정칙성 이론을 포함하는 비국소작용소의 정칙성 이론을 정립함으로써 두 이론을 통합한다는 것이다.

마지막 논문에서는 함수적 차수의 커널을 갖는 선형 비국소작용소에 대하여 경계에서의 정칙성을 연구한다. 갱신함수를 통해 디리클레 문제의 해가 경계 근처에서 어떤 식으로 행동하는지를 분석한다.

주요어휘: 정칙성, 비국소작용소, 확률 과정

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